Quantum Field Theory II
Homework 4 Solutions

1 Strong Interactions with Symmetric Tensors

Your crazy friend Jim (based on a real person) has a particle physics model where, in addition to the six known fundamental-representation quark flavors, there are new quark species which are in the 6 (symmetric tensor or $\square\!\square$) representation. In his model, these have a mass of a few GeV. You are trying to figure out how such particles would change QCD, in order to persuade Jim that they are unlikely to be experimentally viable.

1. What effect would N_6 such quark species have on the beta function of QCD (at one loop or lowest order)? How many species can there be before asymptotic freedom is lost?

2. What would be the simplest and lightest new hadronic bound states containing one of these new quarks Q? What would the expected spin and isospin be?

3. Suppose these new hadrons are to carry an integer electric charge. What are the allowed electric charges of the new Q particles? What is the smallest possible contribution to R the ratio of hadronic to muonic states in e^+e^- annihilation, above the Q mass threshold?

For extra points, see if you can find any literature or other source which presents experimental limits on such particles.

1.1 Solution

- The beta function of QCD at 1 loop is:

$$\mu\frac{d}{d\mu}g = \frac{g^3}{16\pi^2}\left(-\frac{11}{3}C(A) + \sum_f \frac{4}{3}C(f)\right) \tag{1}$$

where the sum runs over all quark species – normal and new. For standard QCD, $C(f) = 1/2$ and we get $-11 + 2N_f/3$ which famously is negative up to $N_f = 16$. But the 6 representation has $C(f) = 5/2$. Therefore we now have $-11 + 2n_f/3 + 10n_{f'}/3$ where $n_{f'}$ is the number of new quarks. So each new quark has the same "weight" as 5 standard quarks. Therefore, 1 or 2 new quarks preserve asymptotic freedom, but 3 new quarks would ruin asymptotic freedom.

- The new quark can couple to two antiquarks which combine into an anti-6 color representation. That's color-symmetric, so something else must be antisymmetric. The lowest energy is the antisymmetric spin-0, so the flavor must be symmetric – isospin 1. So we expect a spin 1/2 isospin-triplet set of states: $Q\bar{u}\bar{u}, Q(\bar{u}\bar{d} + \bar{d}\bar{u})/\sqrt{2}, Q\bar{d}\bar{d}$.

- We see that the new-quark will be partnered with $\bar{d}\bar{d}$, with $\bar{d}\bar{u}$ or with $\bar{u}\bar{u}$, which have charge $+2/3$, $-1/3$, and $-4/3$ respectively. So it needs to have a charge of 1/3 plus an integer: $Q = (1/3) + n$.

  The contribution to R is $Q^2 d_f$ and $d_f = 6$ is the dimension of the representation. So if $Q = 1/3$ then the contribution is $2/3$, whereas for $Q = -2/3$ it would be $8/3$. These are twice as large as the contributions of the b-quark and of the c-quark respectively.

See `https://arxiv.org/pdf/1204.1119.pdf` for experimental limits, at around 500 GeV. The PDG lists a bound on q_6 of 84 GeV. The paper `https://arxiv.org/abs/2005.02512` more-or-less solves this problem for general representations.

2 Beta function to the UV

The strong coupling in the 5-quark MSbar scheme at the scale $\mu = 91$ GeV is $\alpha_s = 0.118$. Use the 1-loop beta function to determine the value of α_s at $\mu = 173$ GeV, the top quark mass.

At this scale, switch to the 6-quark scheme to include top quark effects. Evolve the coupling to the scale $\mu = 10^{16}$ GeV, where the strong coupling may take the same value as the weak coupling and the properly rescaled hypercharge coupling.

2.1 Solution

The one-loop beta function of QCD is

$$\begin{aligned} \frac{\mu^2\, d}{d\mu^2} g^2 &= -\left(11 - \frac{2}{3} N_f\right) \frac{g^4}{16\pi^2} \\ \frac{d(1/g^2)}{d(\ln \mu^2)} & \frac{1}{16\pi^2} \left(11 - \frac{2}{3} N_f\right) . \end{aligned} \tag{2}$$

We see in the second line that it's easier to write the beta function in terms of $1/g^2$ since this eliminates g^2 on the righthand side. The solution is a straight line with slope $(11 - 2N_f/3)/16\pi^2$.

First let's check the factors of 2. If $\alpha_s = 0.118 = g^2/4\pi$ then $1/g^2 = 1/(4\pi \times 0.118)$ or $1/g^2 = 0.674$. This value will reach zero when we change $\ln(\mu^2)$ by $0.674 * 16\pi^2/(11 - 2N_f/3) = 13.9$. Therefore the coupling hits zero at $\Lambda = 91 \exp(-13.9/2) = 0.087$ GeV or 87 MeV. This is smaller than the 300 MeV we were expecting because we are including 5 quark species all the way down. But the point is that it isn't orders of magnitude off, so we got the 2s right.

The value of $1/g^2$ at the scale μ is then:

$$g^{-2}(\mu) = 0.674 + \frac{11 - 2N_f/3}{8\pi^2} \ln \frac{\mu}{91} \tag{3}$$

where it's $1/8\pi^2$ because I am using $\ln(\mu/\mu_0)$ rather than $\ln(\mu^2/\mu_0^2)$. Using $N_f = 5$ and $\mu = 173$ we find $g^{-2} = 0.737$ or $\alpha_s = g^2/4\pi = 0.108$. Starting here and going to $\mu = 10^{16}$ with $N_f = 6$ gives $g^{-2} = 3.546$ or $\alpha_s = 0.0224$ or $1/\alpha_s = 44.6$. For comparison, the weak coupling at the scale 91 GeV is about $1/\alpha_w = 30$. So at this huge scale, the strong coupling may not be the strongest coupling any more.

3 quark-ghost scattering

Calculate the initial-state spin averaged, final-state spin summed squared matrix element for a quark scattering against a ghost. Look up the Feynman rules and remember that the ghost is a spin-scalar and in the adjoint representation. Express your answer in terms of Mandelstamm variables. Remember that there is a minus sign due to the ghost loop which means that your result is actually negative.

3.1 Solution

The color factor averaged over initial colors is

$$
\begin{aligned}
CF &= \frac{1}{d_F d_A} \,\mathrm{Tr}\,_F T^A T^B \;\; \mathrm{Tr}\,_A T^A T^B \\
&= \frac{1}{d_F d_A} C[F]\delta^{AB} C[A]\delta^{AB} \\
&= \frac{1}{d_F d_A} d_A \frac{C_2[F] d_F}{d_A} C_2[A] \\
&= \frac{C_2[F] C_2[A]}{d_A} = \frac{4}{3}\frac{3}{8} = \frac{1}{2}.
\end{aligned} \tag{4}
$$

The matrix element squared averaged over initial spins for initial momenta p, p' and final momenta k, k' is

$$
\begin{aligned}
|\overline{\mathcal{M}}|^2 &= \frac{g^4}{2((p-p')^2)^2} \,\mathrm{Tr}\, \not{p}\not{p}'\not{k}\not{p}' \\
&= \frac{4g^4}{t^2} p \cdot p' \; k \cdot p' = g^4 \frac{su}{t^2}\,.
\end{aligned} \tag{5}
$$

Here I took the Feynman rule for the gluon-ghost vertex, which gives p'_ν, and contracted it with the γ_μ and the $\eta^{\mu\nu}$ from the propagator to write it as $\not{p}'$. An ordinary scalar would have $(p' + k')$ where this has p'. This would lead to $su \to -4su$.

Our answer is negative so we don't need another minus sign. Multiplying by the group theory factor gives $g^4 su/2t^2$. But including antighosts provides another factor of 2. It's fine if you missed this subtlety.

4 Momentum fraction in gluons

This problem takes some work, but maybe it's worth it.

Consider the Altarelli-Parisi equations of QCD, Peskin Eq.(17.128) to (17.130). Assume that four flavors of quark are light and are relevant in our discussion (*udsc*).

It has been observed that about half of the energy/momentum of a proton is carried by the gluons and about half by the quarks and antiquarks. Is this reasonable? Should we expect gluon radiation to turn the proton into purely glue at high resolution scale Q, or do the evolution equations favor a balance between quarks and gluons?

To investigate this, define the total fraction of the proton's energy which is carried by gluons to be

$$X_g \equiv \int_0^1 x\, f_g(x,Q)\, dx = \int_0^1 x\, g(x,\mu)\, dx \tag{6}$$

following either Peskin's notation or the more common one where the PDF is written as $g(x,\mu)$. Similarly, the energy fraction in quarks and antiquarks is

$$X_q \equiv \int_0^1 x \left(\sum_f f_{q_f}(x,Q) + f_{\overline{q}_f}(x,Q) \right) dx = \int_0^1 x \left(\sum_f q_f(x,\mu) + \overline{q}_f(x,\mu) \right) dx \,. \tag{7}$$

Here the sum is over quark flavors. Note that X_g, X_q are functions of Q (or μ depending on notation). We want to explore this scale dependence.

Use the Altarelli-Parisi equations to determine $dX_g/d(\ln(Q))$ and $dX_q/d(\ln(Q))$. Show that $dX_g/d(\ln(Q)) + dX_q/d(\ln(Q)) = 0$ which is conservation of energy. Which one grows and which one shrinks, or does the answer depend on the values of X_g and X_q? Do you find that there is a value of X_g for which the momentum fraction is stable with scale? Is it a UV attractor?

(These quantities X_g and X_q are called the *first Mellin moments* of the PDFs. One can define Mellin moments with any nonnegative integer power of x and analyze their evolution, and for some problems they are simpler and more directly relevant than the PDFs themselves. The moment with x^0 counts the total number of partons; it grows without limit as Q is increased, indeed I think its Q-derivative contains log divergences – total parton number is not well defined. The x^3 moment for quarks is relevant for neutrino scattering on nucleons; these moments are finite and decrease with increasing Q....)

4.1 Solution

First let's write the general form of the DGLAP equations: call the species $f_a(x,\mu^2)$ and the splitting functions $P_{f_a\leftarrow f_b}(x,\mu^2)$. The DGLAP equations are:

$$\frac{\mu^2\partial}{\partial\mu^2}f_a(x,\mu^2)=\sum_{f_b}\int_x^1\frac{dz}{z}f_b(x/z,\mu^2)P_{f_a\leftarrow f_b}(z,\mu^2)\,. \tag{8}$$

We have defined

$$X_a(\mu^2)\equiv\int_0^1 f_a(x,\mu^2)\,x\,dx \tag{9}$$

and we want to know each

$$\begin{aligned}
\frac{\mu^2\partial}{\partial\mu^2}X_a(\mu^2) &= \sum_b\int_0^1 x\,dx\frac{\partial}{\partial\mu^2}f_a(x,\mu^2) && \text{Definition in Eq. (9)}\\
&= \sum_b\int_0^1 x\,dx\int_x^1\frac{dz}{z}f_b(x/z,\mu^2)P_{f_a\leftarrow f_b}(z,\mu^2) && \text{Used Eq. (8)}\\
&= \sum_b\int_0^1 dz\int_0^z dx\;\frac{x}{z}f_b(x/z,\mu^2)P_{f_a\leftarrow f_b}(z,\mu^2) && \text{Reordered integrals}\\
&= \sum_b\int_0^1 dz\int_0^1 z\;dy\;y f_b(y,\mu^2)P_{f_a\leftarrow f_b}(z,\mu^2) && \text{Introduced } y=x/z\\
&= \sum_b X_b(\mu^2)\int_0^1 dz\;zP_{f_a\leftarrow f_b}(z,\mu^2) && \text{Definition of } X_b\\
&\equiv \sum_b C_{ab}X_b(\mu^2)\,,\\
C_{ab} &\equiv \int_0^1 dz\;zP_{f_a\leftarrow f_b}(z,\mu^2)\,.
\end{aligned} \tag{10}$$

All we have to do is to evaluate these coefficients C_{ab}. Energy conservation is obtained if $\sum_a X'(a)=0=\sum_{ab}C_{ab}X_b$ which is true independent of X_b only if $\sum_a C_{ab}=0$. Note that the way we have written the equations, the sum over a should include each flavor and should count quark and antiquark separately.

Now let's find the coefficients explicitly:

$$C_{qq} = \frac{4}{3}\frac{\alpha_s}{2\pi}\int_0^1 dz\, z\left[\frac{3}{2}\delta(1-z) + \frac{1+z^2}{(1-z)_+}\right]$$
$$= \frac{4}{3}\frac{\alpha_s}{2\pi}\int_0^1 dz\, z\left[\frac{3}{2}\delta(1-z) + \frac{2}{(1-z)_+} - (z+1)\right] \tag{11}$$

I have written

$$\frac{z^2}{(1-z)_+} = \frac{1+z^2-1}{(1-z)_+} = \frac{1}{(1-z)_+} - \frac{(1-z)(1+z)}{(1-z)_+} = \frac{1}{(1-z)_+} - (1+z) \tag{12}$$

where the last expression is nonsingular so the + subscript is irrelevant. That way we only need to perform one integral involving this funny subscript:

$$\int_0^1 dz\, z\frac{1}{(1-z)_+} \equiv \int_0^1 dz \frac{z-1}{(1-z)} = -1 \tag{13}$$

With this integral in hand, the other integrals are simple:

$$C_{qq} = \frac{4}{3}\frac{\alpha_s}{2\pi}\left[\frac{3}{2} - 2 - \frac{5}{6}\right] = \frac{4}{3}\frac{\alpha_s}{2\pi}\left[\frac{-4}{3}\right]. \tag{14}$$

Meanwhile,

$$C_{gq} = \frac{4}{3}\frac{\alpha_s}{2\pi}\int_0^1 dz\, z\left[\frac{1+(1-z)^2}{z}\right] = \frac{4}{3}\frac{\alpha_s}{2\pi}\left[1+\frac{1}{3}\right] = -C_{qq} \tag{15}$$

Therefore $\sum_a C_{aq} = 0$ as expected.

Next consider the gluons:

$$C_{qg} = \frac{\alpha_s}{2\pi}\int_0^1 dz\, z\frac{1}{2}\left[z^2 + (1-z)^2\right] = \frac{\alpha_s}{2\pi}\frac{1}{2}\left[\frac{1}{4}+\frac{1}{12}\right] = \frac{1}{6}\frac{\alpha_s}{2\pi} \tag{16}$$

which must be summed (eventually) over quark+antiquark and n_f quark flavors: $1/6 \to n_f/3$. Similarly, the C_{gg} coefficient has two parts: the part from n_f,

$$C_{gg1} = \frac{\alpha_s}{2\pi}\int_0^1 dz\, z\left[-\frac{n_f}{3}\delta(1-z)\right] = -\frac{n_f}{3}\frac{\alpha_s}{2\pi} \tag{17}$$

which will cancel with $\sum_{q,\bar{q}} C_{qg}$, and

$$\begin{aligned} C_{gg2} &= \frac{\alpha_s}{2\pi}\int_0^1 dz\, z\left[\frac{11}{2}\delta(1-z)+6\frac{z^2+(1-z)^2+z^2(1-z)^2}{z(1-z)_+}\right] \\ &= \frac{\alpha_s}{2\pi}\left(\frac{11}{2}+\int_0^1 dz\, z\, 6\left[\frac{1-z}{z}+z(1-z)+\frac{z}{(1-z)_+}\right]\right) \\ &= \frac{\alpha_s}{2\pi}\left(\frac{11}{2}+6\left[\frac{1}{2}+\frac{1}{12}-\frac{3}{2}\right]\right) \\ &= 0 \end{aligned} \tag{18}$$

In summary, for *each* quark and antiquark species, we have

$$\frac{\mu^2 d}{d\mu^2}X_q(\mu^2) = C_{qq}X_q + C_{qg}X_g = \frac{\alpha_s}{2\pi}\left(\frac{1}{6}X_g - \frac{16}{9}X_q\right) . \tag{19}$$

If we write $X_Q = \sum_f X_{q_f} + X_{\bar{q}_f}$ this becomes

$$\frac{\mu^2 d}{d\mu^2}X_Q(\mu^2) = \frac{\alpha_s}{2\pi}\left(\frac{n_f}{3}X_g - \frac{16}{9}X_Q\right) \tag{20}$$

Similarly

$$\frac{\mu^2 d}{d\mu^2}X_g(\mu^2) = C_{gg}X_g + \sum_f C_{gq_f}X_q = \frac{\alpha_s}{2\pi}\left(-\frac{n_f}{3}X_g + \frac{16}{9}X_Q\right) . \tag{21}$$

Clearly $X_Q + X_g$ does not change, since their evolution involves the same combination with opposite coefficients.

The individual energy fractions stay the same if $n_f/3X_g = 16/9X_Q$ which occurs when the gluons have a total energy fraction of $(16/9)/(16/9 + n_f/3)$ which for 4 species is $16/9/(16/9 + 4/3) = 4/7$ so the DGLAP evolution is driven towards the point where the gluons have 4/7 of the energy and the quarks have 3/7.