Renormalization: remaining topics

Anomalous Dimension: suppose for some reason $\beta_y = 0$, y const, so γ_ϕ, γ_ψ constant

Solution to $\frac{M \partial Z}{Z \partial M} = 2\gamma \Rightarrow \frac{dZ}{Z} = 2\gamma \frac{dM}{M}$. $\ln Z/Z_0 = 2\gamma \ln M/M_0$

$$Z = Z_0 (M/M_0)^{2\gamma}$$

2γ is the power of M by which Z grows.

$\gamma > 0$: UV value of Z large (or if $Z_{UV} = 1$, $Z_{IR} < 1$)

(why? we found $\langle\varphi\varphi\rangle = \frac{1}{p^2(1+\gamma \ln M^2/p^2)}$.

$1 + a \ln \frac{M^2}{p^2}$ is first term in expansion of $\left(\frac{M^2}{p^2}\right)^a = e^{a \ln M^2/p^2}$
$= 1 + a \ln \frac{M^2}{p^2} + \dots$

So really we're saying $\langle\varphi\varphi\rangle = p^{2\gamma - 2} M^{-2\gamma}$ weaker growth than p^{-2}.

Actually any operator's correl. must grow slower than p^{-2}

Recall $\langle \mathcal{O}^\dagger \mathcal{O}\rangle(p) = \int_0^\infty ds\, \rho(s) \frac{i}{p^2 - s}$ with $\rho(s)$ positive spectral function.

$-p^2 \frac{d}{dp^2}$ " $= +\int ds\, \rho(s)\, i \frac{p^2}{(p^2-s)^2}$ for spacelike negative p^2, this is always smaller than the original.

~~Why anom. dimension~~?

Another argument: $p^2 \langle \varphi_0 \varphi_0 \rangle$ at large p: chance = 1 that φ_0 creates any state.

$p^2 \langle \varphi_0 \varphi_0 \rangle$ small p: chance it creates state why "Anomalous Dimension"?

I hand you an operator $\mathcal{O}$, energy dimension Δ.

Dimensional grounds: $\langle \mathcal{O}(0)\, \mathcal{O}(x) \rangle \propto x^{-2\Delta}$

Fourier: $\underbrace{\int d^4x}_{x^4}\, e^{ip\cdot x} \underbrace{\langle \mathcal{O}(0)\mathcal{O}(x)\rangle}_{x^{-2\Delta}} \propto p^{2\Delta-4}$ dimensional grounds.

$\propto p^{2\Delta - D}$ (in D spacetime dim)

$\langle \varphi\varphi \rangle = p^{-2}$ because φ is dim-1

$-2 = 2\cdot 1 - D$ ✓

But if φ is really $1+\gamma$-dimensional, expect $\langle \varphi\, \varphi(x)\rangle \propto \dfrac{1}{x^{2(1+\gamma)}}$

and $\langle \varphi\varphi\rangle(p) \propto p^{-2+2\gamma} = p^{2+2\gamma-4}$

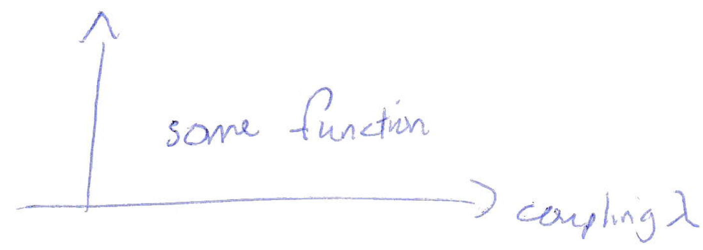
Different scaling with p than p^{-2} same as saying, φ behaves like a field of dim. $1+\gamma_\varphi$

Positivity of spectral function: for gauge-invariant operator, dimension ≥ 1.

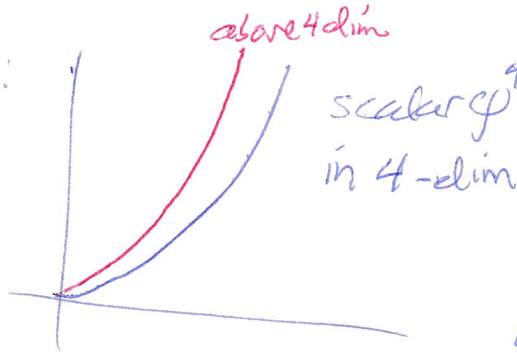
β-functions

$\beta_\lambda = \frac{\mu \partial \lambda}{\partial \mu}$

A theory with 1 coupling has



Interesting cases:

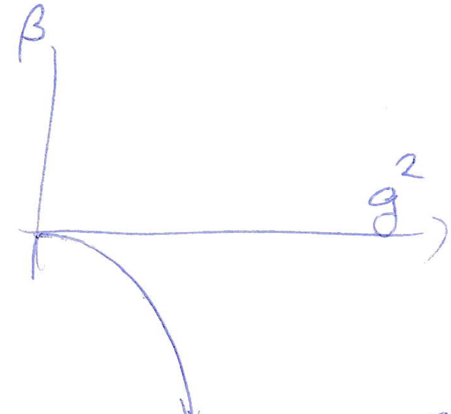


coupling diverges at finite scale.

Theory only makes sense with

1) limiting scale, or

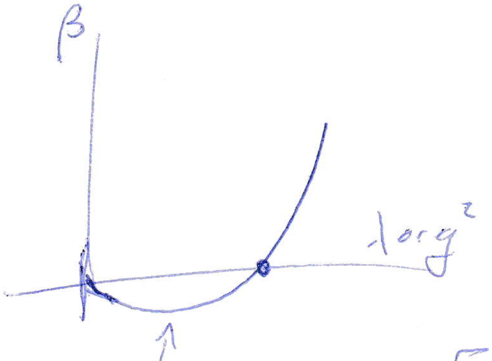
2) coupling = 0 Triviality



coupling → 0 at large scales: "Asymptotic Freedom," theory definitely well defined!!

But coupling diverges at finite IR scale Λ. Something must fundamentally change at this scale – expect different DOF below.

Example: QCD (with few quarks)

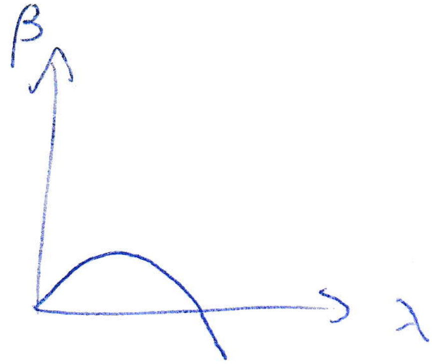


Coupling goes → 0 in UV

but → finite value in IR.

Interacting conformal IR fixed point.

$\lambda\varphi^4$ below 4D (Wilson-Fischer)



Interacting UV behavior goes to free IR behavior.

QCD in 5-dimensions???

No known examples but would be extremely interesting.
Conjecture (Asymptotic Safety) that Gravity behaves like this.

Note, interacting conformal behavior very strange.
All fields & operators have finite anomalous dimensions.
Nothing you can call a "particle"

Also note - when coupling $g^2 \neq 0$, exact Value is scheme dependent.
Location of interacting fixed-pt scheme dependent. Existence not.

Near $\lambda \to 0$: λ^2 and λ^3 (g^3 and g^5, 1 2 loop) terms in β-func. Scheme Independent. Higher terms dependent.

$\overline{MS}$ β-funcs often known to very high loop order, eg.
QCD: 5 loops
$\lambda \varphi^4$: at least 5 loops

Multi-coupling theories have more nontrivial behavior, eg,

Standard Model: $\beta_{g^2} = -\# g^4$ QCD coupling gets smaller in UV

$\beta_{y^2} = y^2(-\#g^2 + \# y^2)$ top-Yukawa gets smaller at slowing rate, but only due to competition

$\beta = +\lambda^2 - y^4$ Higgs coupling gets smaller, can be

Operators

Suppose I want $\langle \phi^2 \rangle$ or, better, $\langle \phi^2_{(x)} \phi^2_{(0)} \rangle_{conn}$

Nobody stops me from adding to $\mathcal{L}$, currents for such op's

$$\mathcal{L} = (\text{usual}) + J_1 \varphi^2 + J_2 \bar{\Psi}\psi + J_3^\mu \bar{\Psi}\gamma_\mu \psi + J_4^\mu \varphi \partial_\mu \varphi + \ldots$$

whatever you want!

Z, W now func's of $J_1 J_2 J_3 J_4$

Can define $\phi^2_{cl} = -\dfrac{\partial W}{\partial J_1}$ etc and include in Legendre transform

Claim: renormalization of such op's not necessarily same as renorm of the "pieces". $J_1 (Z_{\varphi^2} \varphi_0^2) \ldots$ Z_{φ^2} need not $= Z_\varphi \ldots$

Leading calc. of $\langle \phi^2(x) \phi^2(0) \rangle$: $G^2(x-0)$

(note, $G(x-x)G(0-0)$ a disconnected contribution)

consider 3-pt correlator – finite.

$\int d^4x\, e^{ip\cdot x} \langle \phi^2(x) \phi^2(0) \rangle = G^{\phi^2\phi^2}(p)$ mom. sp.

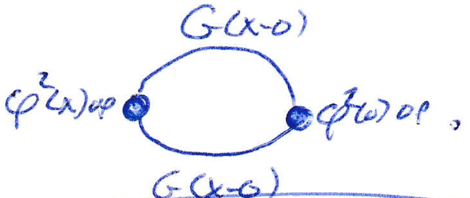
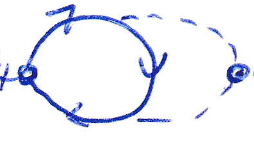
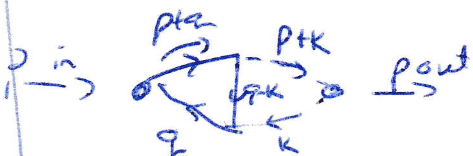


diagram $= \displaystyle\int \frac{d^4q}{(2\pi)^4} \frac{1}{(p+q)^2 q^2}$

NLO : naively $\langle \bar{\Psi}\psi^{(x)} \varphi^{2(0)} \rangle = 0$ but actually $\bar{\Psi}\psi$  φ^2 not zero!



$$\int \frac{d^4q\, d^4k}{(2\pi)^8} \mathrm{Tr}\left(\frac{\not{p}+\not{q}+m}{(p+q)^2-m^2} \frac{(\not{q}-\not{k}+m)}{(q-k)^2-m^2} \frac{\not{q}+m}{q^2-m^2} \right) \frac{1}{(p+k)^2 k^2}$$

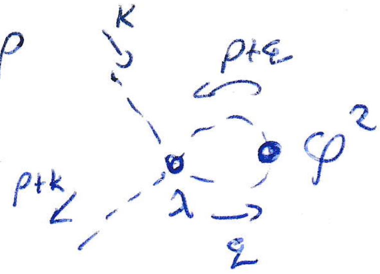
K-integral nicely finite.

Consider $q \gg k$. Answer $\propto m$ vanishes if $m \neq 0$

UV behavior $m \displaystyle\int \frac{d^4q\, q^2}{(q^2)^3}$ log divergent

φ^2 inserts w. finite coeff. on $\rightarrow$ line

1-loop



φ^2 loop correction to φ^2-insertion on φ-line

$$1 \qquad -\lambda \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 (q+p)^2}$$

$$= 1 - \frac{\lambda}{16\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{p^2} + \text{const}\right)$$

I need 1) a prescription for normalization of φ^2-op.

Choose for instance that insertion on φ-line w. momentum $p^2 = -M^2$ has coeff'ct 1.

2) Scale dependence emerges! Need counterterm, and anomalous dim. of φ^2-op also nonzero.

Above diagram: $\frac{\lambda}{16\pi^2} = \gamma_{\varphi^2}$. But γ_φ has no λ contribution. This is new!

 $\bar{\Psi}\psi$ op. we saw $\to \frac{m y^2}{16\pi^2} \# \left(\ln\frac{m^2}{p^2} + \frac{1}{\varepsilon}\right)$. Also  log-div.

$$M \frac{\partial \varphi^3_{op}}{\partial M} = \frac{\lambda}{16\pi^2} \varphi^3_{op}$$

$$M \frac{\partial \bar{\Psi}\psi_{op}}{\partial M} = \frac{\# y^2}{16\pi^2} m \varphi^2_{op} + \frac{\#' y^2}{16\pi^2} \bar{\Psi}\psi_{op}$$

Anomalous Dimension for operators is a matrix! All op's with same ~~charges~~ Q-#s (spin, symmetries) can mix.

High-dim ops can mix in low-dim but not vice-versa.

Conserved currents $\bar{\Psi}\gamma^\mu\psi$, $T^{\mu\nu} = (\bar{\Psi}\gamma^\mu\partial^\nu\psi + \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}(\ldots))$