Representation Theory

$$\psi = \begin{bmatrix} \psi_r \\ \psi_g \\ \psi_b \end{bmatrix} \xrightarrow{V \in SU(3)} \begin{bmatrix} V_{rr} & V_{rg} & V_{rb} \\ V_{gr} & V_{gg} & V_{gb} \\ V_{br} & V_{bg} & V_{bb} \end{bmatrix} \begin{bmatrix} \psi_r \\ \psi_g \\ \psi_b \end{bmatrix} = \begin{bmatrix} \psi_r' \\ \psi_g' \\ \psi_b' \end{bmatrix}$$

$$\psi_a \to V_a{}^b \psi_b$$

$$\bar{\psi} = [\psi_{\bar{r}} \ \psi_{\bar{g}} \ \psi_{\bar{b}}] \longrightarrow [\quad][V^\dagger]$$

$$\bar{\psi}^{\bar{a}} \to \bar{\psi}^{\bar{b}} V^{\dagger}{}_{\bar{b}}{}^{\bar{a}}$$

$$\psi^{\bar{a}} \to \underbrace{(V^\dagger)^T_{ab}}_{V^*} \psi_b$$

$$\underline{A_\mu^A \to A_\mu^A + f_{ABC}\, \theta^B A_\mu^C}$$

Group $\underline{G}$ (of matrices)

Representation of G is set of matrices, one for each group element $g \in G$ $\quad M(g)$

$$\underset{\text{matrixmult}}{M(g_1) * M(g_2)} = M(\underset{\text{grup product}}{g_1 \circ g_2})$$

$SU(2)$: $M(g) = g$ $\quad 2\times 2$ matrix Fund. Rep.

$M(g) = \mathbb{1}$ $\quad 1 \times 1$ $\quad \underset{\mathbb{1}}{M(g_1)}\,\underset{\mathbb{1}}{M(g_2)} = \underset{\mathbb{1}}{M(g_1 g_2)}$ Trivial Rep.

$M(g)$ $\quad 3\times 3$ rotation matrices

$g = \exp i\theta_i \cdot \frac{\tau_i}{2}$ spin 1 $\quad M(g) = \exp \theta_i T^i$

$\quad\;\; \| \\ \quad Q(g)$

$$T^i \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{c} y \\ \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c} x \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{array}$$

Two fields SU(2) ψ, χ Fund.

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad \chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}$$

$\psi \hookrightarrow V\psi$ $\quad \chi \hookrightarrow V\chi$

$V \in SU(2)$

$$V = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

How does $\psi\chi$ behave? Rep. of SU(2)

$$a^*a + b^*b = 1$$

$$\begin{bmatrix} \psi_1\chi_1 \\ \psi_2\chi_1 \\ \psi_1\chi_2 \\ \psi_2\chi_2 \end{bmatrix} = \begin{bmatrix} aa & ab & ba & bb \\ -ab^* & aa^* & -bb^* & ba^* \\ -b^*a & -b^*b & a^*a & ba^* \\ b^*b^* & -ab^* & -b^*a^* & a^*a^* \end{bmatrix} \begin{bmatrix} \psi_1\chi_1 \\ \psi_2\chi_1 \\ \psi_1\chi_2 \\ \psi_2\chi_2 \end{bmatrix}$$

$\psi_a \chi_b \rightarrow V_{aa'} V_{bb'} \psi_{a'} \chi_{b'}$

$$\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

(New?) (4×4 matrix) – Reducible Rep.

clever basis

$$\begin{bmatrix}\psi_1\chi_1\\ \psi_2\chi_1\\ \psi_1\chi_2\\ \psi_2\chi_2\end{bmatrix} \longrightarrow \begin{bmatrix}\psi_1\chi_1\\ (\psi_1\chi_2+\psi_2\chi_1)/\sqrt{2}\\ \psi_2\chi_2\\ (\psi_1\chi_2-\psi_2\chi_1)/\sqrt{2}\end{bmatrix}\begin{matrix}\Big\}\,3\\ \\ 1\end{matrix}$$

$$V_{ice} = \begin{bmatrix}1 & i & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\\ -1 & i & 0 & 0\end{bmatrix}$$

$$V \longrightarrow R V R^{-1}$$

Reducible

$$V_{new} = \begin{bmatrix}a^2+a'^2-b^2-b'^2 & i(a^2-a'^2+b^2-b'^2) & -(\dots) & 0\\ \sim & \sim & \sim & 0\\ \sim & \sim & \sim & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

← 3×3 Irred

1×1 Irred

Combine 2 Reps $R_1 \otimes R_2 = I_1 \oplus I_2 \oplus I_3$

$$\left(\tfrac{3}{2}\right) \otimes (1) = \left(\tfrac{5}{2}\right) \oplus \left(\tfrac{3}{2}\right) \oplus \left(\tfrac{1}{2}\right)$$

$$4 \times 3 = 6 + 4 + 2$$

$$\left(M(g_1) \times M(g_2)\right)^* = \left(M(g_1 \circ g_2)\right)^*$$

If I find Rep M.

$$M^*(g_1) \times M^*(g_2) = M^*(g_1 \circ g_2)$$

$\boxed{\text{Automatic} \to M^* \text{ Conjugate Rep.}}$

If $M \in \mathbb{R}^{n \times n} \to$ same. Real 3×3 SU(2)

If Basis choice $RMR^T = M^*$ Pseudoreal 2×2

$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

sometimes, new distinct Rep!

SU(2), Fund. is Pseudoreal

SU(3) & higher, not ψ^* is something new.

Structure constants (N_c^2-1)

$T^{(a)}$ gen. $[T^A, T^B] = i f_{ABC} T^C$

$SU(2): f_{abc} = \epsilon_{ABC}$

$(F^C)_{(AB)} = -i f_{ABC}$

indices "which one"

$$F^1_{(23)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} -$$

$$F^2_{(13)} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} -$$

spin 1

$$F^3_{(12)} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} -$$

$$[F^A, F^B] = i f_{ABC} F^C$$

$$[[T^A,T^B],T^C] + [[T^B,T^C],T^A] + [[T^C,T^A],T^B] = 0$$

$f_{ABD} [T^D, T^C]$

Jacobi Identity

Adjoint Rep

Irred. Reps are described by a few #s invariants

d – dimension of rep.

spin 1/2 2 –
spin 1 3 –
spin 0 1 [0]
$(2s+1)$

$SU(3)$
ψ : $d=3$
A^A_μ : $d=8$

C Trace norm or Dynkin Index

$$\mathrm{tr}\, T^A T^B = D^{AB}$$
$$= C\,\delta^{AB}$$

sensible normal T^A

Choose C for one rep → fixes C for all others

$SU(2)$: $T^A = \frac{\sigma^A}{2}$. $f^{ABC} = \epsilon_{ABC}$

$\mathrm{Tr}\, \frac{\sigma^1}{2}\frac{\sigma^1}{2} = \mathrm{Tr}\begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} = 1/2$

$SU(3)$ $T^A = \lambda^A/2$

$C_F = 1/2$

$\underline{\underline{SU(N):}}$

$d_F = N$

$d_A = \#\text{traceless Herm. } N\times N = N^2-1$

$\boxed{\begin{array}{l} d_T = 1 \\ C_T = 0 \\ C_{2,T} = 0 \end{array}}$

$C_F = 1/2$ $\qquad C_{2,F} = \frac{N^2-1}{2N}$

$C_A = \dots N$ $\qquad C_{2,A} = N$

C_2 second Casimir

$$T^2 = \sum_A T^A T^A = C_2 \mathbb{1}$$

$$[T^2, T^B] = 0 \qquad \text{"like } J^2 \; [J^2, J_i]\text{"}$$

C, d, C_2 $\underline{\text{not indep.}}$

$$\mathrm{Tr}\, T^A T^A = C_2 \mathrm{Tr}\, \mathbb{1} = C_2 d_R$$

$$= C \delta^{AA} = \boxed{d_A C = C_2 d_R} \qquad C_2 = \frac{C d_A}{d_R}$$

Groups have G — Dimension d_A

— Rank : # of mutually commuting Lie elements

$SU(2)$: $\frac{\tau^1}{2}, \frac{\tau^2}{2}, \frac{\tau^3}{2}$ $\quad \left[\frac{\tau^A}{2}, \frac{\tau^B}{2}\right] = i\epsilon_{ABC}\frac{\tau^C}{2}$

$\boxed{\text{Rank } 1}$

$SU(3)$ $\quad \frac{\lambda^1}{2} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \frac{\lambda^8}{2} = \begin{bmatrix} 1/\sqrt{12} & 0 & 0 \\ 0 & 1/\sqrt{12} & 0 \\ 0 & 0 & -2/\sqrt{12} \end{bmatrix}$ $\quad \left[\frac{\lambda^1}{2}, \frac{\lambda^8}{2}\right] = 0$

Rank 2

$SU(N)$ - $N-1$ diagonal gen's $\qquad$ Rank $N-1$

What groups can show up? — can be internal symm?

Coleman Mandula: Symm of QFT are

Transl + $SO(3,1)$ space-time symm Poincaré

$\otimes$ internal symm group G → Compact Lie $\times$ discrete stuff

G = semi-simple group

$U(1) \times U(1) \times \underbrace{G_1 \times G_2 \times}_{\text{simple}}$

What are all possible simple Groups?

Group	Rank	Dimension
$SU(N)$	$N-1$	N^2-1
$SO(2N)$	N	$\frac{2N(2N-1)}{2}$
$SO(2N+1)$	N	$\frac{(2N+1)(2N)}{2}$
$Sp(2N)$	N	$N(2N+1)$
G_2	2	14
F_4	4	26
E_6	6	78
E_7	7	133
E_8	8	248

$SU(2)$ -

$SU(3)$ -

3

$SU(2) \simeq SO(3)$

$SP(2) \simeq SU(2)$

$SU(4) = SO(6)$

$SO(4) = SU(2) \otimes SU(2)$

3

exceptional

Gut $SU(5)$
$SO(10)$
E_6

String $E_8 \otimes E_8$
$SO(32)$

What are all the Reps ?? $SU(N)$

Always have trivial

fund.

adj.

others? yes, but build by $\otimes$ these

$SU(N)$ $\epsilon_{i_1 \dots i_N} \to$ invariant N fund. fields $\psi^1 \dots \psi^N$

$\left[\epsilon_{abcd\dots} \psi^1_a \psi^2_b \dots \psi^N_z\right] \xrightarrow{V}$ itself

$SU(2)$: $\underbrace{\psi_1 \chi_2 - \psi_2 \chi_1} = \epsilon_{ab} \psi_a \chi_b$

$SU(3)$ $\underbrace{\epsilon_{abc} \psi_a \chi_b \sigma_c}$ staysame

2 fund. ψ χ

$\epsilon_{ab\dots} \psi_a \chi_b$ irred rep.

symm. also irred.

SU(2): ψ, χ

$$\left.\begin{array}{c} \psi_1\chi_1 \\ \psi_1\chi_2 + \psi_2\chi_1 \\ \psi_2\chi_2 \end{array}\right\} \text{spin-1}$$

$\psi_1\chi_2 - \psi_2\chi_1$ — spin 0

SU(3) ψ, χ

σ

$$\left.\begin{array}{l} \sigma_3(\psi_1\chi_2 - \psi_2\chi_1) \\ -\sigma_2(\psi_1\chi_3 - \psi_3\chi_1) \\ +\sigma_1(\psi_2\chi_3 - \psi_3\chi_2) \end{array}\right\} \text{Rep } \bar{3}$$

$$\left.\begin{array}{l} \psi_1\chi_1 \quad \psi_2\chi_2 \quad \psi_3\chi_3 \\ \psi_1\chi_2 + \psi_2\chi_1 \\ \vdots \end{array}\right\} 6 \text{ Rep}$$

SU(N) $\Big\{$ N fields $\in \ldots \longrightarrow$ singlet

N−1 fields $\in \ldots \longrightarrow \bar{N}$

M antisymm comb in $SU(N)$: $\binom{N}{M} = \frac{N!}{M!(N-M)!}$ ways

M symm comb " $\frac{(N+M-1)!}{M!}$

Young's Tableau - how to do Symm/Antisymm

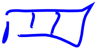
 find ψ_a

-  antisymm $\psi_{[a}\chi_{b]}$ $\qquad \frac{N(N-1)}{2}$

 symm $\psi_{\{a}\chi_{b\}}$ $\qquad \frac{N(N+1)}{2}$ $\Big\} N^2$

- //  $\psi_{[a}\chi_{b}\sigma_{c]}$ $\qquad \frac{N(N-1)(N-2)}{6}$

for N=3
N=4 - antif.



 $\psi_{(a}\chi_{b)}\sigma_c$

SU(2)

$2 = N$

1 $\quad \frac{N(N-1)}{2}$

3 $\quad \frac{N(N+1)}{2}$

2

4

28

$$\frac{2 \cdot 3 \cdot 4 \cdot 5}{2 \cdot 3 \cdot 4} \cdots$$

$10 \otimes 3$

$= 15' + 15$

$$2 \otimes 2 = 1 \oplus 3$$

SU(3) $= 3$

$$= \frac{3 \cdot 2}{2} = \bar{3}$$

$= 6$ $\quad = 10$ $\quad = 15$

$= 8$ Adj $\quad = 1$

$$3 \otimes 3 = \bar{3} \oplus 6$$

$$\bar{3} \otimes 3 = 1 \oplus 8$$

$$\alpha \otimes \beta = \gamma \oplus \delta \oplus \epsilon$$

$$\sum C = C_\gamma + C_\delta + C_\epsilon = d_\beta C_\alpha + d_\alpha C_\beta$$

SU(3)

$$3 \otimes \bar{3} = 1 \oplus 8$$

$$3 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = 0 + C_8$$

$$\boxed{C_A = 3}$$

$$3 \otimes 3 = \bar{3} \oplus 6$$

$$3 = \tfrac{1}{2} + \tfrac{5}{2}$$

$$\boxed{C_6 = 5/2}$$