Yang-Mills (Nonabelian Gauge) Thy

1) Define it
2) Build path $\int$ for it

Define it: Write Path Integral

$$\int \mathcal{D}A_\mu \underbrace{\mathcal{D}(\bar\psi\psi)}_{\text{if spinors}} \underbrace{\mathcal{D}(\varphi^\dagger\varphi)}_{\text{if scalars}} \exp iS[A,\bar\psi\psi\,\varphi^\dagger\varphi]$$

$S =$ last lecture: $\frac{1}{4g_0^2} F_{0\mu\nu}^{A} F_0^{\mu\nu A} + \bar\psi_0(i\not{D}-m_0)\psi_0 + \dots$

(Bare fields)

Does this make sense?

- Same as Q: Is there a regulator where I can define this?

Perturbatively - yes, $\overline{MS}$. We're coming to that

Nonperturbatively - yes, Lattice - in Euclidean metric, Minkowski is $\underset{p^0 \to -ip^0}{t \to it}$ continuation as discussed.

Lattice: 1) Really we want to declare

$$\begin{aligned} &A_\mu(x) && A_\mu^{A\prime} = A_\mu^A + D_\mu\theta^A \\ &\psi(x) \text{ and } && \psi' = (1+i\theta^A T^A)\psi \\ &\varphi(x) && \varphi' = \quad '' \quad \varphi \end{aligned}$$

as equivalent, and $\int$ only over equivalence classes. Well defined but technically super hard.

Only correl. func's of gauge-invariant operators, eg. $\bar\psi\gamma^\mu\psi$, $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\alpha}F^{\alpha\beta}$ etc. ~~well~~ nonzero.

LW14P2

Not ready to just use lattice for everything
- Analytic contin. is hard - time domain?
- UV behavior?

We should also build Perturb. Thy, see what it can teach us.

$\overline{MS}$ is compatible with nonab. gauge invariance ('t Hooft Veltman 1972 NPB 44, 189 >4700 cites)
unlike, eg, cutoff, Pauli-Villars, etc

Problem: $\int \mathcal{D}A_\mu$ measure includes arbitrarily large A_μ.

Controlled by action $e^{-S[A]}$. Scalar: $\int \mathcal{D}\varphi\, e^{-\int(\frac{|\nabla\varphi|^2}{2}+m^2\varphi^2)}$

φ can be big

Gaußian convergence factor

Usually this is enough.

But not for A^μ, since we know $A_\mu \propto \partial_\mu \theta$ has $S=0$
no matter how big $\partial_\mu\theta$ is.

"Flat directions" in integral.

Solved on lattice by discrete space - θ "orbit" finite
Finite integration.
But not for us.

Need to do something to prevent large $A_\mu \propto \partial_\mu\theta$

Same approach as QED: gauge fixing

Pick function which $\underline{\underline{\text{does}}}$ depend on gauge:

$$G[A] : G^A(x) = \partial^\mu A^A_\mu(x) - \underbrace{k^A(x)}_{\text{added in for later convenience.}}$$

Requirement: $G^A[A'(x)] = 0$ at all A, x

has $\underline{\text{one}}$ unique solution over all (canonical) gauge choices

– Does it? $A^A_\mu \to A^A_\mu + \partial_\mu \theta^A - g f_{ABC} \theta_B A^C_\mu$

$$\to \partial^\mu A_\mu \to \partial^\mu A_\mu + \partial^\mu \partial_\mu \theta - \underline{g f_{ABC}\left[\partial^\mu \theta_B A^C_\mu + \theta_B \partial^\mu A^C_\mu\right]}$$

If I assume second term $\underline{\underline{\text{smaller}}}$,

I can pick $\theta = \frac{-1}{\partial_\mu \partial^\mu}(\partial^\mu A_\nu)$ approximately solves, and then iterate.

Requires $\underline{\underline{\text{assuming}}}$ $g A_\mu$ "small."

Perturbation theory – that's $\underline{\text{true}}$ except when $gA \sim 1$

and $F_{\mu\nu} F^{\mu\nu} \sim \frac{1}{g^2} \to e^{-S} \sim e^{-1/g^2}$

exponentially suppressed by $\exp[-1/\alpha]$ beyond scope of Pert. Thy.

- Within P.T., it's a valid self-consistent assumption.

[But is it true? No – Gribov found counterexamples, all involving large $A_\mu \sim 1/g$, $\int F^2 \sim 1/g^2$ ... skip for now]

Start with

$$\int \mathcal{D}A^\mu \, \mathcal{D}(\bar\psi \psi \text{ etc}) \, e^{iS[A,\bar\psi,\psi]}$$

can I just put $\delta(G^A(A(x)))$ in integral? Not obvious. (and - no you can't)

<u>But</u> $\underbrace{\int \mathcal{D}\Theta^A}_{\text{all gauge choices}} \; \delta(G^A(\underbrace{A_\Theta}_{A_\mu \text{ after } \Theta\text{-transform, eg, } A_\mu + \partial_\mu \Theta})) \; \underbrace{\text{Det} \frac{\delta G^A[A_\Theta]}{\delta \Theta^B}}_{\text{Jacobian}} = 1$

Recall, $\int dx \; \delta(f(x)) \neq 1$. But $\int dx \; \delta(f(x)) \underbrace{\left|\frac{\partial f}{\partial x}\right|}_{\text{Jacobian}} \stackrel{\text{is}}{=} 1$

~~That's why I need~~

$\int dx\, dy \; \delta(f(x,y))\, \delta(g(x,y))$ also $\neq 1$

but put in $\left| \text{Det} \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} \right|$ and it <u>is</u>!
Same idea, just infinite $\int$'s.

Exchange integrals

$$\int \mathcal{D}\Theta^A \int \mathcal{D}A^A_\mu \, \mathcal{D}(\bar\psi\psi..) \, e^{iS[A,\bar\psi\psi]} \; \underbrace{\delta[G^A[A(x)]]}_{\text{one } \delta \text{ for each } x \text{ for each } A} \; \text{Det} \frac{\delta G^A[A_\Theta]}{\delta \Theta^B}$$

Here

$$G^A[A(x)] = \partial^\mu A^A_\mu(x) - \alpha^A$$

$$G^A[A_\theta] = \partial^\mu A^A_\mu(x) + \partial^\mu [D_\mu \theta(x)]^A$$

$$\text{"} \; \partial_\mu \theta^A - g f_{ABC} \theta^B A^C_\mu$$

(ADJOINT rep. cov. deriv)

so $$\frac{\delta G^A[A_\theta]}{\delta \theta^B} = [\partial^\mu D_\mu]^A{}_B$$

$$= \delta^{AB} \partial^\mu \partial_\mu - g f_{ABC} \partial^\mu A^C_\mu$$

Since α^A was anything, integrate w. Gaußian weight

$$\int \mathcal{D}\alpha \exp \frac{i}{\xi} \int d^Dx \frac{\alpha^2(x)}{2} \int \mathcal{D}\theta \int \mathcal{D}A_\mu \, e^{iS[A]} \, \delta(\underbrace{\partial_\mu A^\mu_\theta - \alpha}_{=G}) \, \mathrm{Det}\, \partial_\mu D^\mu_\theta$$

Change variables in $\int \mathcal{D}A_\mu \to \int \mathcal{D}A^{-\theta}_\mu$ reverses θ-transform

$(A^\mu_{-\theta})_\theta = A^\mu$ takes θ out of δ-func.

$\mathrm{Det}\, \partial_\mu D^\mu_\theta$

$S[A_{-\theta}] = S[A] = S[A_\theta]$ as action is gauge-invariant.

No θ-dependence left - $\int \mathcal{D}\theta$ is harmless overall factor

$$\int \mathcal{D}\alpha \, e^{\frac{i}{2\xi}\int \alpha^2} \int \mathcal{D}A_\mu \, e^{iS[A]} \, \delta(\partial_\mu A^\mu - \alpha) \, \mathrm{Det}[\partial_\mu D^\mu]$$

Use δ-func. to perform α-integral. $\alpha \to \partial_\mu A^\mu$

$$\int \mathcal{D}A_\mu \exp i \int \left(\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right) \mathrm{Det}[\partial_\mu D^\mu]$$

Gauge fixing just like in Abelian. Except Fadeev-Popov determinant

For QED, $D^\mu = \partial^\mu - \cancel{f_{ABC}\,\theta_B A^\mu_C}$

$\qquad 0 \quad$ same as ∂_μ in adjoint, eg, γ not charged

Det $\partial_\mu \partial^\mu$ is A-independent – another boring constant!

QCD: Det $\partial^\mu D_\mu$ is not boring constant.

↳ really (x,y) space, (AB)-space matrix.

$$[\partial^\mu D_\mu]^{AB}_{(x,y)} = \delta^{AB}\partial^\mu_x \partial_\mu\, \delta^4(x-y)$$
$$\qquad - f_{ABC}\,\partial^\mu_x A^C_\mu(x)\, \delta^4(x-y)$$

Determinant is A_μ-dependent.
You cannot ignore it. What to do?

Represent determinant w. Grassmann Integral

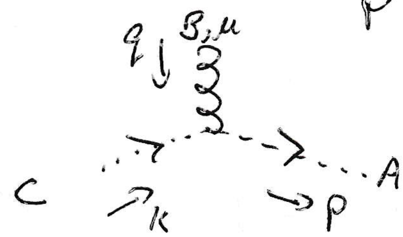
$$\mathrm{Det}\, M_{ab} = \int \mathcal{D}\bar{c}_a c_b \exp -i\,\bar{c}_a M_{ab} c_b$$

Adj. valued Grassmann, but not spinors.

Scalars, Grassmann? "Ghosts" (wrong spin-statistics)

$$Z_{QCD} = \int \mathcal{D}(A_\mu \bar{\psi}\psi\, \bar{c}c) \exp i \int \mathcal{L} + \frac{1}{2\xi}(\partial_\mu A^\mu_A)(\partial^\nu A_{\nu A}) - \bar{c}\,\partial_\mu D^\mu c$$

c complex field.  $\dfrac{i}{p^2+i\varepsilon}$



$-g f_{ABC}\, p^\mu$ not $(k^\mu + p^\mu)$ just p^μ

 ghost loop – extra – sign (Grassmann)