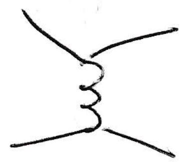
β-function of QCD / YM / Nonab. Gauge

Consider YM, group G, fermions: N_f species in rep. R.

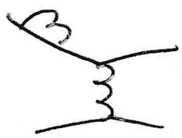
$g^2(\mu)$ μ-dependent. $\beta_g \equiv \mu \frac{\partial g}{\partial \mu}$ What's its value?

Consider Scattering:

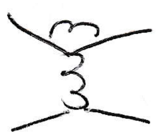


LO: $g^2 \times$ (something)

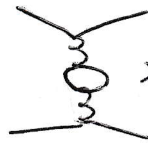
NLO:



$\times 4$,

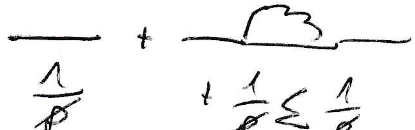


$\times 2$,



$\times 1$ but

$\mathcal{M}$ is Amputated correlator using correctly normalized external states. If



$$\frac{1}{\not p} + \frac{1}{\not p}\Sigma\frac{1}{\not p} = \frac{1}{\not p - \Sigma} = \frac{Z}{\not p}$$

$Z = 1 + (\not p \text{ part of } \Sigma)$

then $Z^{-1/2}\psi$ creates states

so $\mathcal{M} = Z^{-2} \times$(Amputated diagram)

$$= g^2 + g^4 \times \left[\frac{1}{\varepsilon} + \log\frac{\mu^2}{p^2}\right] \times \Big(-2 \times \text{coeff of } \not p \text{ in } \Sigma$$

$+ 2 \times$ coeff of $\gamma^\mu T^A$ in

$+ 1 \times$ coeff of $q^2 g^{\mu\nu}$ in

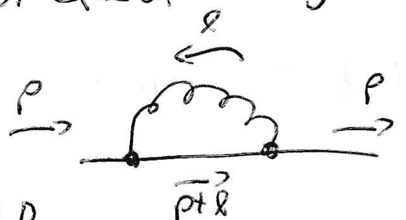
I need to know $\frac{1}{\varepsilon}$ parts of $\not p$ in Σ

$\gamma^\mu T^A$ in

$q^2 g^{\mu\nu}$ in $\Pi^{\mu\nu}$

$$\mu\frac{\partial \mathcal{M}}{\partial \mu} = 0 = 2g\,\mu\frac{\partial g}{\partial \mu} + 2g^4 \times [\text{coeff on } 1/\varepsilon] \quad \text{or} \quad \beta = -g^3 \times \left(\tfrac{1}{\varepsilon}\text{ coeff.}\right)$$

Start with Σ. We already did this for QED. Only difference – group theory

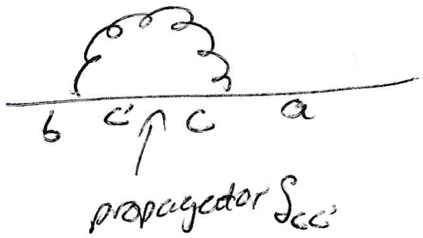


$$\mu^{4-D}\int\frac{d^D l}{(2\pi)^D}\underbrace{(iT^A g\gamma^\mu)}_{\text{vertex}}\underbrace{\frac{i(\not p+\not l)}{(p+l)^2+i\epsilon}}_{\text{propagator}}\underbrace{(iT^B g\gamma^\nu)}_{\text{vertex}}\underbrace{\frac{(-i)\left(g_{\mu\nu}+(\xi-1)\frac{l_\mu l_\nu}{l^2}\right)}{l^2+i\epsilon}}_{\text{gluon propagator}}\delta_{AB}$$

Exactly like QED except $T^A, T^B, \delta_{AB}\cdots$

This "group thy part" factors out from rest of calc.

Incoming color a, outgoing color b



$$\delta_{AB}\,T^A_{ac}\,T^B_{cb}=\underline{T^A_{ac}\,T^A_{cb}}$$

Option 1: write all (N_c^2-1) T matrices, square them, ...

Don't let me ever catch you doing this

Option 2: use what we learned $T^A_{ac}T^A_{cb}=C_2[R]\,\mathbb{1}_{ab}$

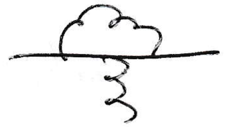
$\Sigma = C_2[R]\,\mathbb{1}_{ab}\times(\text{QED-value})$.

only new factor.

$\mathbb{1}_{ab}$ expected – color preserved.

$\mathbb{1}_{ab}$ $\qquad$ $\mathbb{1}_{ab}\,\mathbb{1}_{bc}\,\mathbb{1}_{cd}=\mathbb{1}_{ad}$

(boring)

Vertex?  in QED this canceled Σ.

Does it still? well, Calc. again identical to QED except extra "color stuff"

~~GTOL~~

Color-part only: $T^C T^A T^B \delta_{BC}$

Option 1: for each A, mult. L, R by each T^B....
Don't ever do this, it's for desperate people

Clever way: If Only $T^C T^B T^A \delta_{BC} = T^B T^B T^A = C_2(R) T^A$

Oh-but $T^A T^B = T^B T^A + [T^A, T^B]$

$$T^B T^A T^B = T^B T^B T^A + T^B [T^A, T^B]$$

$$C_2[R]\, T^A + \underbrace{T^B i f_{ABC} T^C}$$

$$= \frac{1}{2} i f_{ABC} [T^B, T^C] \quad \text{since } f_{ABC} \text{ antisymm.}$$

$$= \frac{1}{2} i f_{ABC} f_{BCD} T^D$$

For SU(2): $\underbrace{\epsilon_{ABC}\epsilon_{BCD}} = \delta_{CC}\delta_{AD} - \delta_{CD}\delta_{AC} = (3-1)\delta_{AD} = 2\delta_{AD}$

Generally - must be (something) $\times\ \delta_{AD}$, no? But what?

$F_{(BC)}{}^{A} = -i f_{BCA}$ Defines Adj. Rep.

$$f_{ABC} f_{BCD} \underset{\substack{\text{antisymm}\\ \text{twice}}}{=} f_{BCA} f_{BCD} = -f_{CBA} f_{BCD} = +F_{(CB)}{}^{A} F_{(BC)}{}^{D}$$

$$= \mathrm{Tr}\, F^A F^D = C[A]\,\delta^{AD}$$

$$\boxed{T^B T^A T^B = \left(C_2[R] - \tfrac{1}{2} C[A]\right) T^A}$$

We computed "everything else"  already.

$C_2[R]$ part still cancels $C_2[R]$ part of Σ

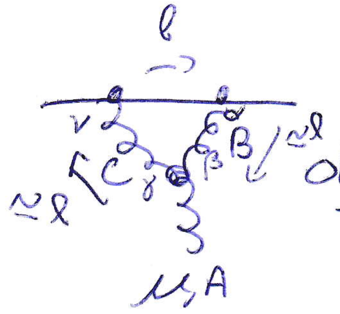
But $C[A]$ uncanceled – it gets left over!

$$\frac{1}{\epsilon} \text{ piece} \longrightarrow -\frac{\xi}{2}\frac{C[A]}{16\pi^2} \times \left(\frac{1}{\epsilon} + \dots\right)$$

Note: Σ, Γ don't cancel

Answer is ξ dependent — troubling – but that's what it is!

Also need



oh boy

keep only large-ℓ asymptotic-enough to find $1/\epsilon$ bit.

$$(ig)^2 \underbrace{T^C T^B g f_{ABC}}_{\text{also } -\frac{1}{2} C_2[A] T_A} \int_\ell \gamma^\alpha \frac{i\ell\!\!/}{\ell^2} \gamma^\nu \frac{(-i)\left(g_{\alpha\beta}+(\xi-1)\frac{\ell_\alpha \ell_\beta}{\ell^2}\right)}{\ell^2}$$

$$\times \frac{(-i)\left(g_{\nu\gamma}+(\xi-1)\frac{\ell_\nu \ell_\gamma}{\ell^2}\right)}{\ell^2}$$

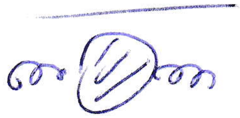
$$\times \left(g_{\mu\gamma}\ell_\beta + g_{\mu\beta}\ell_\gamma - 2 g_{\gamma\beta}\ell_\mu\right)$$

--- work to find $\frac{1}{\ell^4}$ behavior

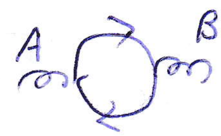
$-\frac{1}{2}\xi$ from other $\delta\Gamma$ term

Algebra: $\Sigma + \delta\Gamma$ contribute $\left(\frac{3g^3}{4}\right)\left(1+\frac{\xi}{3}\right)\frac{C[A]}{16\pi^2}$

Um- ξ did not drop out! $\Sigma, \delta\Gamma$ did not cancel!



Fermion

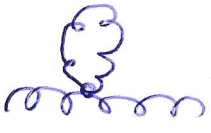


we have seen.

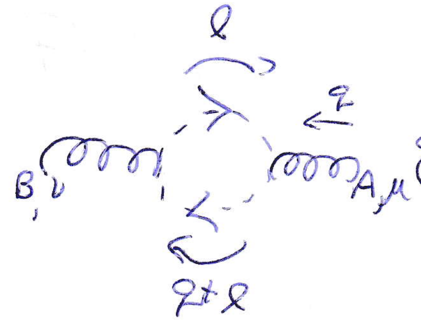
Group: $\mathrm{Tr}\, T^A T^B = C[R]$

so p^2 piece of $\Pi \to -\frac{g^2}{12\pi^2} C[R] N_f$ ← # of fermion species

$\times (p^2 g_{\mu\nu} - p_\mu p_\nu)\frac{1}{\epsilon}\ldots$



$= 0$ in $\overline{MS}$. Whew!



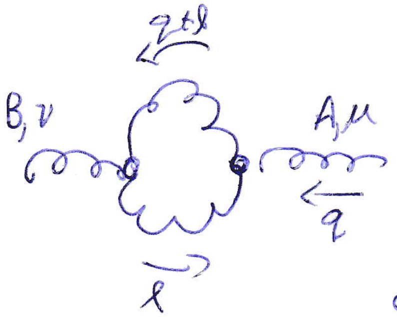
ghost.

$$(-)\, g^2 \int \frac{d^D \ell}{(2\pi)^D}\, \underbrace{f_{ACD}\, f_{BDC}}_{-C[A]}\, \frac{i\ell_\mu\, i(q+\ell)_\nu}{\ell^2 (q+\ell)^2}$$

ghost loop

Work (not hard!) $\frac{ig^2}{16\pi^2}\frac{1}{\varepsilon} C[A]\left(\frac{1}{12} g_{\mu\nu} q^2 + \underline{\frac{1}{6} q_\mu q_\nu}\right)$

eek! Not transverse!



The Bad Boy!

$$g^2\, \underbrace{f_{ACD}\, f_{BCD}}_{C[A]\,\delta_{AB}} \int \frac{d^D\ell}{(2\pi)^D} \times (-i)^2\, \frac{\left(g^{\alpha\gamma} + (\xi-1)\frac{\ell^\alpha \ell^\gamma}{\ell^2}\right)}{\ell^2}\, \frac{\left(g^{\beta\delta} + \dots\right)}{(\ell+q)^2}$$

$$\times \left[g_{\mu\alpha}(q-\ell)_\beta + g_{\alpha\beta}(2\ell+q)_\mu + g_{\mu\beta}(-2q-\ell)_\alpha\right]$$

$$\times \left[g_{\nu\gamma}(\ell-q)_\delta + g_{\gamma\delta}(-2\ell+q)_\nu + g_{\nu\delta}(\ell+2q)_\gamma\right]$$

lots of ugly algebra $\frac{ig^2}{16\pi^2} C[A]\frac{1}{\epsilon}\left[\left(\frac{25}{12}-\frac{\xi}{2}\right) g_{\mu\nu} q^2 - \left(\frac{7}{3}-\frac{\xi}{2}\right) q_\mu q_\nu\right]$

Sum w. ghosts is transverse!

$$\Pi^{\mu\nu}_{tot} = \frac{ig^2}{16\pi^2}\frac{1}{\varepsilon}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[\left(\frac{13}{6}-\frac{\xi}{2}\right)C[A] - \frac{4}{3} N_f C[R]\right]$$

β function $\mu\frac{\partial}{\partial\mu} g = \frac{g^3}{16\pi^2}\left[\frac{4}{3}C[R]N_f - \frac{11}{3}C[A]\right]$

from Π

$$\frac{13}{6} - \frac{\xi}{2} + \underline{\frac{3}{2} + \frac{\xi}{2}}$$

Main Lesson: Unlike $\left\{\begin{array}{l}\lambda\varphi^4\\ \text{Yukawa}\\ \text{QED}\\ \text{Scalar-spinor-QED}\end{array}\right.$

Beta-func. is negative $\underset{=}{\text{if}}$ $\frac{11}{3}C[A] > \frac{4}{3}C[R]N_f$

For QCD: $C[A] = N_c = 3$ $\quad 11 > \frac{2}{3}N_f$

$C[R] = 1/2$ $\quad N_f < 33/2$

≤ 16 quark species – Asympt. Free.

Nature: 6 species.

With scalars: $\frac{4}{3}C[R]$ per Dirac, $\frac{2}{3}C[R]$ per Weyl fermion

$\frac{1}{3}C[R]$ per $\mathbb{C}$, $\frac{1}{6}C[R]$ per real scalar

N=4 SYM thy: 4 Adjoint Weyl fermions

6 " ~~R~~ real scalars

$$-\frac{11}{3}C[A] + 4\cdot\frac{2}{3}C[A] + 6\cdot\frac{1}{6}C[A] = \frac{-11+8+3}{3}C[A] = 0$$

cool.