Anomalies

K 17P1

Definition - Something unexpected - indeed, opposite of what you expect

QFT definition - when a symmetry is present in classical $\mathcal{L}$, but regularization of thy necessarily breaks symm, and it is not recovered in regulariz $\to 0/\infty$ limit

Example : Scale or Conformal Anomaly

Massless thy, eg,

$$\mathcal{L}[A_\mu, \psi, \varphi] = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \pm D_\mu\varphi^* D^\mu\varphi \mp \frac{\lambda}{2}(\varphi^*\varphi)^2$$
$$+ \bar{\psi}(i\not{\partial})\psi - y\varphi\bar{\psi}\psi + \text{h.c.}$$

$x \to \xi x$ so $\partial_x \to \xi^{-1}\partial_x$

$\psi \to \xi^{-3/2}\psi$ $\qquad \int d^4x \to \xi^4 \int d^4x$

$\varphi \to \xi^{-1}\varphi$

$A \to \xi^{-1} A$ $\qquad S = \int d^4x\, \mathcal{L} \to S$ unchanged.

(Or $g_{\mu\nu} \to \xi^2 g_{\mu\nu}$ ...)

Predicts

$T^{\mu\nu}_{(x)} = 2\frac{\delta S}{\delta g_{\mu\nu}(x)}$ traceless

g^2 etc. scale-invariant ($\beta = 0$, $\gamma_\varphi = 0$ etc)

Wrong! Must introduce scale in regularizing / regulating thy.

μ or latt-spacing a or...

Transform corresponds to changing μ/a at fixed bare param.

Not symmetry. ~~A~~ Scale or Conformal Anomaly

"The" Anomaly - Adler Bell Jackiw
Bardeen

Chiral symmetries generically broken in gauge thy

~~Warm-up:~~ "~~Baby~~" ~~version~~

Chiral symmetry: consider $\left(\begin{matrix}1 \\ \text{or} \\ N_f\end{matrix}\right)$ massless spinor ψ_f

$$\mathcal{L} = \bar\psi i \not{D} \psi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

"unusual" symmetry: defining γ_5 as usual $\quad \begin{matrix} P_L \\ P_R \end{matrix} = \begin{matrix} \frac{1-\gamma_5}{2} \\ \frac{1+\gamma_5}{2} \end{matrix} \quad \begin{matrix} P_L^2 = P_L \\ P_R^2 = P_R \\ P_L P_R = 0 \end{matrix}$

Note $\gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5$ so $P_L \gamma_\mu = \gamma_\mu P_R$
$P_R \gamma_\mu = \gamma_\mu P_L$

$$(P_L \psi)^\dagger = \psi^\dagger P_L^\dagger = \psi^\dagger P_L \quad \text{but} \quad \bar\psi = \psi^\dagger \gamma^0$$

$$\bar\psi P_L = \psi^\dagger \gamma^0 P_L = (\psi^\dagger P_R) \gamma^0$$

bar of $P_R \psi$

Massless thy - looks like you can transform $\begin{matrix} P_L \psi \\ P_R \psi \end{matrix}$ separately

1 spinor: $\psi \to (P_L e^{i\theta_L} + P_R e^{i\theta_R}) \psi$

$$\psi^\dagger \to \psi^\dagger (P_L e^{-i\theta_L} + P_R e^{-i\theta_R})$$

$$\bar\psi \to \bar\psi (P_R e^{-i\theta_L} + P_L e^{-i\theta_R})$$

$$\bar\psi \gamma^\mu \psi \to \bar\psi (P_R e^{-i\theta_L} + P_L e^{-i\theta_R}) \gamma^\mu (e^{i\theta_L} P_L + e^{i\theta_R} P_R) \psi$$

$$= \bar\psi \gamma^\mu (P_L e^{-i\theta_L} + P_R e^{-i\theta_R})(P_L e^{i\theta_L} + P_R e^{i\theta_R}) \psi$$

$$= \bar\psi \gamma^\mu \psi \quad \text{so } \mathcal{L} \text{ unchanged.}$$

N_f spinors: replace $e^{i\theta_L} \to V_L \in SU(N_f) \times U_1(\theta) = U(N_f)$

1 spinor: Expect two conserved currents

$$j_L^\mu = \bar{\psi}\gamma^\mu P_L \psi \quad \text{from } P_L e^{i\theta_L} \text{ trans.}$$

$$j_R^\mu = \bar{\psi}\gamma^\mu P_R \psi \quad \text{from } P_R e^{i\theta_R} \text{ trans.}$$

or equivalently: $P_L e^{i\theta_L} + P_R e^{i\theta_R} = (e^{i\theta_V}) e^{i\gamma^5\theta_A}$

(infintesimally $1 + i\theta_L P_L + i\theta_R P_R = 1 + i\theta_V + i\theta_A \gamma_5$

$\theta_V = (\theta_L + \theta_R)/2$

$\theta_A = (\theta_R - \theta_L)/2$)

N_f spinors: $2(N_f^2 - 1)$ more currents

$$j_{LA}^\mu = \bar{\psi}\gamma^\mu P_L T^A \psi$$

$$j_{RA}^\mu = \bar{\psi}\gamma^\mu P_R T^A \psi$$

$j_{VA}^\mu = (j_L + j_R)/\cancel{2}$

$j_{AA}^\mu = (j_{RA}^\mu - j_{LA}^\mu)/\cancel{2}$

Fact: in presence of gauge int, these are not all present!

j_V^μ's remain conserved, as do j_{VA}^μ j_{AA}^μ

j_A^μ does not - Axial / Chiral / ABJ Anomaly

why? Totally not obvious, so we do it slowly

Baby Version: 2D QED

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{2} F_{01}^{2}$$

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (= \sigma_2)$$

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (= i\sigma_1)$$

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (= \sigma_3)$$

Only 1 Lorentz transform - boost along x-axis

$J^{\mu\nu} = J^{01}$. Trans. on ψ: $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$$S^{01} = \frac{i}{4}[\gamma^0, \gamma^1] = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Lorentz does not mix upper & lower entries.

Define $\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$ $\quad \bar{\psi} = [\psi_+^* \ \psi_-^*]\gamma^0 = [i\psi_-^* \ -i\psi_+^*]$

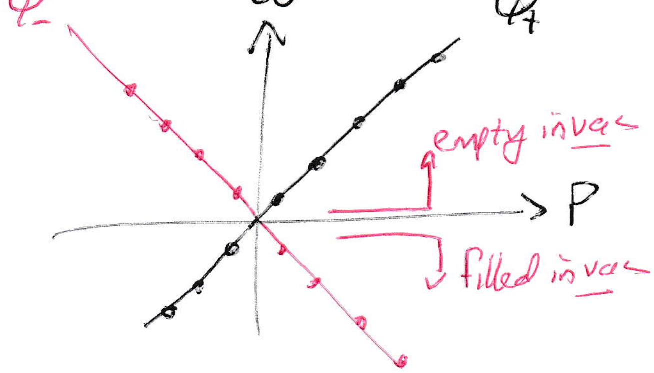
$i\not{\partial}$ again switches L, R - ψ_+^* only couples to ψ_+
ψ_-^* " " " ψ_-

Thy logically consistent (??) w. only one of the two!!

Dirac Eq $\quad i(\partial_0 + \partial_1)\psi_+ = 0 \quad$ sol'n $\psi_+(x,t) = \psi_+(x-t)$
Right mover $e^{i(px-\omega t)}$ $p=\omega$

$i(\partial_0 - \partial_1)\psi_- = 0 \quad$ $\psi_- = \psi_-(x+t)\, e^{i(px-\omega t)}$
Left-mover $p = -\omega$

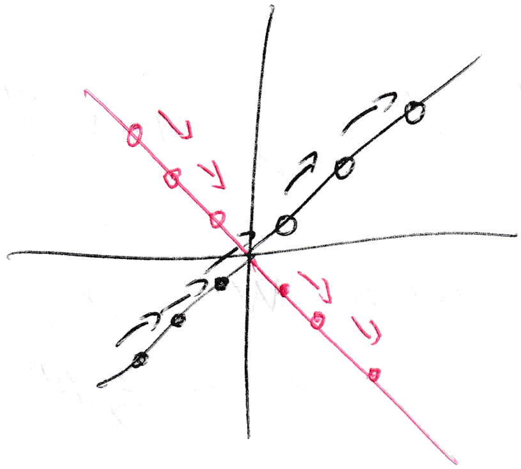
In box of length L, allowed p-values quantized: $\frac{2\pi}{L}$ spacing



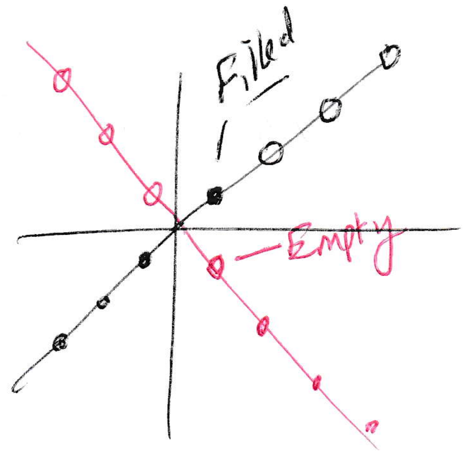
Turn on E-field. ~~Kinetic~~

ψ_+ gain energy & p $\Big\}$ at rate $\frac{dp}{dt} = eE$ $\quad \frac{d\mathcal{E}}{dt} = \begin{cases} eE \\ -eE \end{cases}$

ψ_- gain p, lose energy



after time t with $\int E\,dt = \frac{2\pi}{Le}$ slide 1 "spot"



one R-particle
one L-antiparticle
were created.

Therefore $\int dx\,dt\, j_R^0 = 1$

Note, $\int dx\,dt\, E = L\int E\,dt = L\left(\frac{2\pi}{Le}\right) = \frac{2\pi}{e}$

$$\partial_0 j_R^0 = \frac{eE}{2\pi} = \frac{e}{4\pi}\underbrace{\epsilon_{\mu\nu}F^{\mu\nu}}_{\text{fancy way to write } E}$$

$$\boxed{\partial_\mu j_R^\mu = -\partial_\mu j_L^\mu = \frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}}$$

If particles ψ_+, ψ_- have same electric chg: $j_{em}^\mu = j_R^\mu + j_L^\mu$

$\partial_\mu j_{em}^\mu = 0.$

If $q_+ = q_- + \bar{q}$, $\quad \partial_\mu j_{em}^\mu = \bar{q}\frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}$

Thy with $\bar{q} \neq 0$, or where ψ_- doesn't exist $\xrightarrow{\text{A Problem}}$ Inconsistent.

Good way to see it's true

I want $j_R^\mu = \bar\psi(x)\gamma^\mu\underbrace{\left(\frac{1+\gamma^5}{2}\right)}_{=P_R}\psi(x)$

But dangerous to have $\bar\psi(x), \psi(x)$ at 1 point. Singular lim. of correl. func.

Must mean $\lim_{\varepsilon^\mu\to 0} \bar\psi(x+\frac{\varepsilon^\alpha}{2})\gamma^\mu P_R \quad \psi(x-\frac{\varepsilon^\alpha}{2}) = j^\mu$

oops, need $W(x+\frac{\varepsilon}{2}, x-\frac{\varepsilon}{2})$
$\simeq (1-ie\varepsilon_\alpha A^\alpha)$

Now take $\partial_\mu j^\mu$: Deriv acts on $\bar\psi$, on ψ, AND on $\varepsilon_\alpha A^\alpha$

~~$\partial_\mu\bar\psi = (D_\mu + ieA_\mu)\bar\psi$ as $D_\mu\psi = (\partial_\mu + ieA_\mu)\psi$~~

~~$\partial_\mu\psi = (D_\mu - ieA_\mu)\psi$ $\quad D_\mu\bar\psi = \ " \ " \ " \ \bar\psi$~~

~~$\gamma^\mu D_\mu\psi = 0$ so $\gamma^\mu\partial_\mu\psi = ieA\psi$ BUT $\gamma^\mu P_R D_\mu\psi = P_L\gamma^\mu D_\mu\psi$ careful~~

$$\partial_\mu \lim_{\varepsilon_\alpha\to 0}\left[\bar\psi(x+\tfrac{\varepsilon}{2})\gamma^\mu P_R(1-ie\varepsilon_\alpha A^\alpha)\psi(x-\varepsilon/2)\right]$$

Term 1) $\partial_\mu\bar\psi(x+\varepsilon/2)\gamma^\mu P_R$ write $\partial_\mu = \partial_\mu - ieA_\mu + ieA_\mu = D_\mu + ieA_\mu$

$D_\mu\bar\psi\gamma^\mu = 0$ Dirac Eq. $\rightarrow \quad ieA_\mu^{(x+\varepsilon/2)}\bar\psi\gamma^\mu P_R\ldots$

Term 2) $\ldots\gamma^\mu P_R\partial_\mu\psi = P_L\gamma^\mu\partial_\mu\psi = -ieA_\mu P_L\gamma^\mu\psi = -ieA_\mu^{(x-\varepsilon/2)}\gamma^\mu P_R\psi$

Sum, term 1 + term 2: $ie\underbrace{(A_\mu(x+\varepsilon/2) - A_\mu(x-\varepsilon/2))}\bar\psi\gamma^\mu P_R\psi$

$ie\varepsilon^\nu\partial_\nu A_\mu\bar\psi\gamma^\mu P_R\psi$

Term 3: $\frac{\partial}{\partial x^\mu} \bar{\Psi} \gamma^\mu P_R (1 - ie\,\varepsilon_\nu A^\nu(x))\psi$

$\cancel{-ie\,\varepsilon_\nu \bar{\psi}\gamma^\mu}\ -ie\,\varepsilon^\nu \partial_\mu A_\nu \bar{\Psi}\gamma^\mu P_R \psi$

$$\partial_\mu J_R^\mu = ie(\partial_\nu A_\mu - \partial_\mu A_\nu)\bar{\Psi}\gamma^\mu \varepsilon^\nu P_R \psi$$

At short distance ~~$\langle\psi\bar{\psi}\rangle$~~ $\langle \psi(x - \varepsilon/2)\bar{\Psi}(x+\varepsilon/2)\rangle = \frac{-i}{2\pi}\frac{\gamma^\alpha \varepsilon_\alpha}{\varepsilon^2}$

(Real-space Fermion Propagator in 1+1 dim, Fourier of $\frac{i\not{p}}{p^2}$.)

$$\partial_\mu J_R^\mu = \lim_{\varepsilon\to 0} \frac{(ie)(-i)}{2\pi} F_{\nu\mu} \,\mathrm{Tr}\, \frac{\gamma^\alpha \varepsilon_\alpha \varepsilon_\nu \gamma^\mu P_R}{\varepsilon^2}$$

$\lim_{\varepsilon\to 0} \frac{\varepsilon_\alpha \varepsilon_\nu}{\varepsilon^2} = \frac{g_{\nu\alpha}}{2}$ if I take $\varepsilon \to 0$ avg over directions

$\mathrm{Tr}\,\gamma^\alpha\gamma^\mu = g^{\alpha\mu}\,\mathrm{Tr}\,\mathbb{1} = 2g^{\alpha\mu}$ ~~Oops I~~ $\to 0$ $\qquad P_R = \frac{1+\gamma_5}{2}$

$\mathrm{Tr}\,\gamma^\alpha\gamma^\mu\gamma_5 = 2\epsilon^{\alpha\mu}$ will not $\to 0$, I get $\frac{(ie)(-i)}{2\pi} F_{\mu\nu} \frac{1}{2} \frac{g_{\alpha\nu}}{\cancel{2}} \epsilon^{\alpha\mu} \cancel{2}$

$$\partial_\mu J_R^\mu = -\partial_\mu J_L^\mu = \frac{1}{4\pi} F_{\mu\nu}\varepsilon^{\mu\nu} \quad \text{Not } 0$$

as $P_L = \frac{1-\gamma_5}{2}$

Repeat in 4D

Suppose $E_x \neq 0$ And $B_x \neq 0$

Energies set by Landau levels of B:

Density of states in lowest Landau level $= \frac{eB}{2\pi}$

These states act like 1D system: $\partial_\mu j_R^\mu = \frac{e^2}{4\pi^2}\vec{E}\cdot\vec{B}$ ....

$$\boxed{\partial_\mu j^\mu = e^2 \,\epsilon^{\mu\nu\alpha\beta} F\, F}$$

Or, repeat point-splitting argument

Note, $\langle \psi(y)\bar{\psi}(z)\rangle = \frac{-i}{2\pi^2}\frac{\gamma^\alpha(y-z)_\alpha}{(y-z)^4}$ cubic divergent must work harder

Also $\mathrm{Tr}\,\gamma^\mu\gamma^\alpha\gamma^5 = 0$ but $\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5 = \frac{4i}{\cancel{24}}\epsilon^{\mu\nu\alpha\beta}$

More work - see Peskin - but in the end

$$\partial_\mu J^\mu_R = \frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$$

Yang-Mills: $e^2 \to g^2 C[R]$, $F^A_{\mu\nu}F^A_{\alpha\beta}$

diff # of L, R handed charged species, $\sum_{L-R} g^2 C[R] \neq 0$

If $J^\mu_{R\,\mathrm{or}\,L\,\mathrm{or}\ldots}$ is intended to be current of gauge field, and it is not conserved, thy is garbage (or, doesn't exist - no consistent regularization.)

But can't I always regulate it with the lattice?

NO! Nielsen-Ninomiya

NPB 195 (1982) 541

NPB 193 (1981) 173

PhysLett B 105 (1981) 219