Consider the London-Anderson-Brout-Englert-Higgs-Guralnik-Kibble-Hagen Mechanism (Higgs for short)

Start w QED w scalar field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi^* D^\mu\varphi - V(\varphi^*\varphi)$$

$$[(\partial_\mu + ieA_\mu)\varphi^*][(\partial^\mu - ieA^\mu)\varphi]$$

$$V(\varphi^*\varphi) = \lambda(\varphi^*\varphi)^2 - \mu^2\varphi^*\varphi + \frac{1}{4}\frac{\mu^4}{\lambda}$$
$$= \frac{\lambda}{4}(2\varphi^*\varphi - v^2)^2$$

$v^2 = \mu^2/\lambda$ Vac. Value

Classical energy minimum

$$\varphi = \frac{v}{\sqrt{2}} \qquad \left(\varphi = \frac{\varphi_r + i\varphi_i}{\sqrt{2}} \quad \varphi_r = v\right)$$

Equivalent vacua $\frac{v}{\sqrt{2}}e^{i\theta}$ no longer phys. distinct - gauge transforms of $v/\sqrt{2}$

W.O. A_μ: phase-fluctuations are massless:

$\varphi = \frac{v}{\sqrt{2}}e^{i\theta(x,t)}$ has $\partial_t^2\theta = \partial_x^2\theta$ wave Eq with $m=0$

$\varphi = \frac{(v+h)}{\sqrt{2}}$ has $\partial_t^2 h = \partial_x^2 h - m^2 h$ massive with $m^2 = 2\mu^2$

But with A^μ? Fluctuations in $A_\perp$ eg, if $\partial_z\theta \neq ik(\theta)$

then $A_x(z), A_y(z)$

Fluct. in A_z, A_0, φ_r (or h), φ_i (or θ)

One of these can be removed by gauge fixing!

Think first about $A_\perp$:

Higgs 2

Suppose $A_x(z,t) = c\,e^{i(kz-\omega t)}$

k fixed
ω to be determined

$\varphi = \frac{v+h+ig}{\sqrt{2}}$ also $\sim e^{i(kz-\omega t)}$

$\vec{D}\cdot\vec{D}\varphi = D_z D_z \varphi$

$\vec{D}\cdot\vec{D} = \partial_x^2 + \underbrace{2ieA_x\partial_x}_{0 \text{ as } \partial_x \to 0} + \mathcal{O}(A_x^2)$

no "source" for φ: $h=0=g$ consistent.

$\frac{\delta \mathcal{L}}{\delta A_x}$: $\partial_z^2 A_x$ from $F_{\mu\nu}F^{\mu\nu}$

$\partial_t^2 A_x$ " "

And: $[(\partial_x + ieA_x)\varphi^*][(\partial_x - ieA_x)\varphi] \to ie[\varphi^* D_x\varphi - \varphi D_x \varphi^*]$

But this is not 0: $\varphi^* = \frac{v}{\sqrt{2}}$, $\varphi = \frac{v}{\sqrt{2}}$, $\varphi^* D_x\varphi \to -ieA_x\frac{v^2}{2}$

$\varphi D_x \varphi^* \to ieA_x v^2/2$

$\to e^2 v^2 A_x$

$\partial_t^2 A_x = \partial_z^2 A_x - \underbrace{e^2 v^2 A_x}_{\text{from } D_\mu\varphi^* D^\mu\varphi \text{ term}}$

Massive dispersion, mass $m = ev = e\sqrt{2\varphi^*\varphi}$

Interpretation: When $\varphi \neq 0$,

making $A_x \neq 0$ costs energy as $D_x\varphi \neq 0$

Gauge bosons are now massive!

Higgs 3

Extend to h, g, A_z, A_t

Think first about plain-old-E&M

$$\mathcal{L} = -F_{\mu\nu}F^{\mu\nu} = (-\partial_t A_i + \partial_i A_t)^2 - (\partial_i A_j - \partial_j A_i)^2$$

$$\partial^\mu F_{\mu\nu} = 0 = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu$$

$\nu = x$ $\quad -\partial_z^2 A_x + \partial_t^2 A_x - \underbrace{\cancel{\partial_t \partial_x A_t}}_{0} + \underbrace{\cancel{\partial_z \partial_x A_z}}_{0} = 0 \to (k^2 - \omega^2) A_x = 0$

$\partial_x = 0 = \partial_y$

$\partial_z = ik, \; \partial_t = -i\omega$

similarly $(k^2 - \omega^2) A_y = 0$

$\nu = z$: $\quad \partial_t^2 A_z - \cancel{\partial_z^2 A_z} - \partial_t \partial_z A_t + \cancel{\partial_z \partial_z A_z} = 0$

$$-\omega^2 A_z - \omega k A_t = 0 \qquad \text{mixes } A_z, A_t$$

$\nu = t$: $\quad \cancel{\partial_t^2 A_t} - \partial_z^2 A_t - \cancel{\partial_t \partial_t A_t} + \partial_z \partial_t A_z = 0$

$$k^2 A_t + \omega k A_z = 0$$

$$\begin{bmatrix} k^2-\omega^2 & 0 & 0 & 0 \\ 0 & k^2-\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & -k\omega \\ 0 & 0 & k\omega & k^2 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \\ A_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ if $\omega^2 = k^2$ ✓ $\quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ if $\omega^2 = k^2$ ✓ $\quad \begin{bmatrix} k \\ -\omega \end{bmatrix}$ always, eg, $A_z = -i\partial_z \theta$

$A_t = -i\partial_t \theta$

Pure Gauge.

And $[\omega]$ if $\omega^2 + k^2 = 0$

Pure QED

Transverse solutions for $\omega^2 = k^2$

Pure-gauge solution always present

No final solution $\to$ only 2 polarizations.

Now Add Scalar Field!

$$\partial^\mu F_{\mu\nu} + j_\nu = 0 \qquad ie(\varphi^* D_\mu \varphi - \varphi D_\mu \varphi^*)$$

and $$D^\mu D_\mu \varphi + \frac{\partial V}{\partial \varphi^*} = 0$$

$$D_\mu \varphi = (\partial_\mu - ieA_\mu)\left(\frac{v + h + ig}{\sqrt{2}}\right) = \frac{-ieA_\mu v + \partial_\mu h + i\partial_\mu g}{\sqrt{2}}$$

$$D^\mu D_\mu \varphi = (\partial^\mu - ieA^\mu) \quad = \frac{-ie(\partial_t A_t - \partial_z A_z)v + \partial_\mu \partial^\mu h + i\partial_\mu \partial^\mu g}{\sqrt{2}}$$

$$ie(\varphi^* D_\mu \varphi - \varphi D_\mu \varphi^*) = \frac{(iev)}{\sqrt{2}}\big(-ievA_\mu + \partial_\mu h + i\partial_\mu g$$
$$-ievA_\mu - \partial_\mu h + i\partial_\mu g\big)/\sqrt{2}$$
$$= (ev)^2 A_\mu - ev\,\partial_\mu g \qquad (\text{Call } ev = m)$$

$$\cancel{\varphi^*\varphi = \frac{(v+h)^2}{2} + \frac{g^2}{2} \to 2\text{ order}}$$

$$\frac{\partial V}{\partial \varphi^*} = \frac{2\mu^2 h}{\sqrt{2}} \quad \text{as in pure-scalar theory.}$$

$$\text{Re } D^\mu D_\mu \varphi + \frac{\partial V}{\partial \varphi^*} \to \partial_t^2 h - \partial_z^2 h + 2\mu^2 h = 0, \quad (-\omega^2 + k^2 + 2\mu^2)h = 0$$

$$\text{Im} \quad " \quad \to \partial_t^2 g - \partial_z^2 g - m\partial_t A_t + m\partial_z A_z = 0$$

Put it together

$$\begin{bmatrix} -\omega^2+k^2+m^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega^2+k^2+m^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega^2+m^2 & -k\omega & -ikm & 0 \\ 0 & 0 & k\omega & k^2+m^2 & i\omega m & 0 \\ 0 & 0 & ikm & i\omega m & -\omega^2+k^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega^2+k^2+2\mu^2 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \\ A_t \\ g \\ h \end{bmatrix}$$

A_z, A_t, g mix! Ugh!

Automatic Zero Mode: $\begin{bmatrix} k \\ -\omega \\ -im \end{bmatrix}$ $\quad A \to A + \partial\theta$

$\varphi \to e^{i\theta}\varphi = (v + iev\theta)$

$\delta g = ev\theta = m\theta \ldots$

Physical excitation $\begin{bmatrix} \omega \\ k \\ \frac{2i\omega k}{m} \end{bmatrix}$ if $\omega^2 = k^2+m^2$, eg., EV $-\omega^2+k^2+m^2$

Last state never satisfied as in gauge thy alone

3 Physical modes, $\omega^2 = k^2 + \underbrace{(ev)^2}_{=m_A^2}$ gauge mass

$A_x, A_y, \to A_z$ as k gets small

Higgs mode decouples, $m_h^2 = 2\mu^2$

Complicated because A_z, A_t, "g" Im-part of φ all inter-couple - coupled ODEs.

Better way? At small-fluct. level, yes.

Gauge-fix: choose gauge.

Don't choose $\partial_\mu A^\mu = 0$ as before - instead

If $\varphi = \left(\frac{v+h}{\sqrt{2}} + ig\right)(x)$, gauge rotate at x to eliminate Im component of φ,

$$\varphi \to e^{i\theta}\varphi \quad \theta = -\operatorname{atan}\left(\frac{g}{v+h}\right)$$

Works for $\|g\|, \|h\| \ll v$. Assume for now

$$D_\mu\varphi = (\partial_\mu - ieA_\mu)\left(\frac{v+h}{\sqrt{2}}\right) \to D_\mu\varphi D^\mu\varphi = \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{m^2}{2}A_\mu A^\mu \quad (m = ev)$$

$$\boxed{\mathcal{L} = -F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu + \frac{1}{2}\partial_\mu h \partial^\mu h \mp V(h)}$$

Invertible now! $\langle A_\mu(x) A_\nu(0)\rangle = G_{\mu\nu}(x)$

$$G_{\mu\nu}(p) = \frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)}{p^2 - m^2 + i\epsilon} \quad \text{check!}$$

At tree level you can ignore/discard "g", compute with h: $\langle hh\rangle = \frac{i}{p^2 - m_h^2}$

$$A_\mu \quad G_{\mu\nu} = \frac{-i(g_{\mu\nu} - p_\mu p_\nu/m^2)}{p^2 - m^2 + i\epsilon}$$

and it's all fine