Renormalization

Problem: parameters & fields in $\mathcal{L}$ not what we measure.

Step 1: name everything in sight.

$$\begin{array}{ccccccc} \phi & & \phi_0 & & M^2 & & M_0^2 \\ A & \rightarrow & A_0 & \text{Bare fields} & \lambda & & \lambda_0 \\ \psi & & \psi_0 & & g & \rightarrow & g_0 \quad \text{Bare couplings} \\ & & & & y & & y_0 \\ & & & & m & & m_0 \end{array}$$

$$\mathcal{L} = D_\mu \phi_0^\dagger D^\mu \phi_0 - M_0^2 \phi_0^\dagger \phi_0 - \frac{\lambda_0}{8}(\phi_0^\dagger \phi_0)^2$$

$$+ \bar{\psi}_0 (i\not{D} - m_0)\psi_0 - y_0 \phi_0 \bar{\psi}_0 \psi_0$$

$$+ \frac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})^2 \qquad D_\mu = \partial_\mu - i g_0 T^A A_\mu^A$$

Now name sensibly rescaled versions $\quad Z_\phi \phi_r^2 = \phi_0^2$

$\phi_r, A_r, \psi_r \,;\, \lambda_r, g_r, y_r, M_r^2, m_r \,.\quad Z_\psi \bar{\psi}_r \psi_r = \bar{\psi}_0 \psi_0$

defined in terms of some Renormalization ~~Criteria~~ Conditions

– Approach 1: "on-shell" renormalization

Require renorm. fields & couplings to obey some sensible conditions related to stuff we can measure, eg,

$$\int \langle T[\phi_r(x)\phi_r(0)]\rangle e^{ip\cdot x} d^4x = G_r(p) \text{ has pole at } p^2 = m_r^2$$

with residue $Z = 1$

eg, if I write $G_r(p) = \dfrac{iZ}{p^2 - M^2 + i\epsilon} + (\text{finite})$ then $Z = 1$, $M^2 = M_r^2$

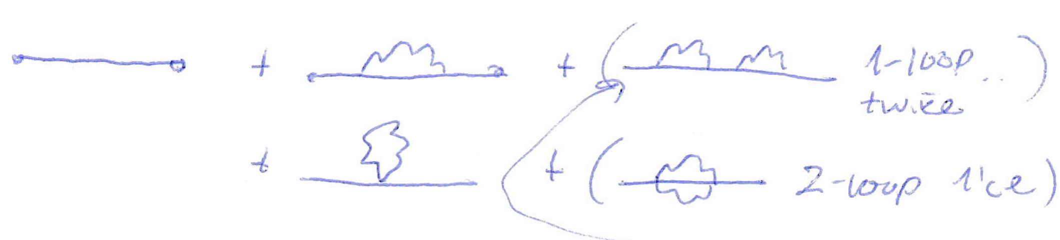
Couplings: some conditions, for instance

e_r : $\frac{e_r^2}{4\pi r}$ = long-distance Coulomb potential

λ_r : something like Re part of $\phi\phi \to \phi\phi$ Scatt amplitude at .... $= \lambda_r$

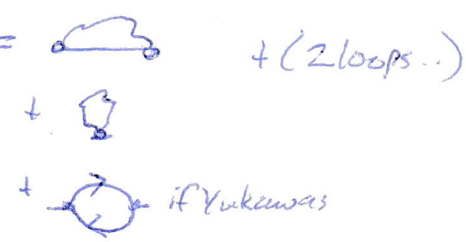
Now imagine computing things at loop-order

$\langle \phi_0 \phi_0 \rangle(p) =$



We already saw, we can incorporate all these

by writing $\langle \phi_0 \phi_0 \rangle = \dfrac{i}{p^2 - M_0^2 - \Pi(p)}$ $\quad \Pi(p) =$



Evaluating $\Pi(p)$, I will find terms of form

$$\Pi(p) = e_0^2\left(\#\left[\ln\left(\frac{\mu^2}{p^2}\right) + \frac{1}{\varepsilon} + \text{const}\right]p^2 + \left(\#'\ln\left(\frac{\mu^2}{p^2}\right) + \frac{1}{\varepsilon} + \text{const}\right)M_0^2\right)$$
$$+ y_0^2\left(\#''\left[\ln\frac{\mu^2}{p^2} + \frac{1}{\varepsilon} + \text{const}''\right]p^2 + \mathcal{O}(e_0^4)\right.$$
$$\left. + \#'''[\;''\quad ''\quad ''\;]M_0^2\right)$$

ϕ_r is a rescaling of ϕ_0: $Z\phi_r^2 = \phi_0^2$

$$\langle \phi_r \phi_r \rangle = \frac{i}{Z\left[p^2 - M_0^2 - (\# e_0^2 + \#'' y_0^2)\left(\ln\frac{\mu^2}{p^2} + \frac{1}{\varepsilon} + \text{const.}\right)p^2 - (\#' e_0^2 + \#''' y_0^2)(\cdots)M_0^2\right]}$$

LL 2P3

Condition 1:

$$1 = \frac{\partial}{\partial p^2}\left(Z \left(p^2 - (\#e_0^2 + \#'' y_0^2)\left(\ln\left(\frac{\mu^2}{p^2}\right) + \frac{1}{\varepsilon} \cdots\right) p^2\right) + \text{const}\right)\Bigg|_{p^2 = M_r^2}$$

$$\cancel{1 = Z - Z(\#e_0^2 + \#'' y_0^2)(-1) \cdot Z = \frac{1}{1+\#e_0^2 + \#'' y_0^2} \quad Z = 1 - \#e_0^2 \#'' y_0^2}$$

$$1 = Z - Z(e_0^2 \# + y_0^2 \#'')\left(\ln + \frac{1}{\varepsilon} + c'\right) + \mathcal{O}(e_0^4)$$

$$Z = \frac{1}{1 + (\#e_0^2 + \#' y_0^2)\left(\ln\frac{\mu^2}{M_r} + \frac{1}{\varepsilon} + c'\right)} \simeq 1 - (\#e_0^2 + \#'' y_0^2)\left(\ln\frac{\mu^2}{M_r^2} + \frac{1}{\varepsilon} + c\right) + \mathcal{O}(e^4)$$

So we can compute Z_ϕ etc.

Similarly $p^2 = M_r^2$ should give pole position

$$0 = \underset{M_r^2}{p^2} - M_0^2 - (\#e_0^2 + \#'' y_0^2)\left(\ln + \frac{1}{\varepsilon}\right) p^{2 \to M_r^2}$$
$$- (\#' e_0^2 + \#''' y_0^2)\left(\ln + \frac{1}{\varepsilon}\right) M_0^2 + \mathcal{O}(\cdot\cdot)$$

$$(1 - \text{first log}) M_r^2 = (1 + \text{second log}) M_0^2 \quad \cdots$$

Similarly $\langle \phi_r \phi_r \phi_r \phi_r \rangle$ gives matrix element $\mathcal{M} = \lambda_r$ at physical pt.

$$\frac{1}{Z^2}\Big(\lambda_0 + \lambda_0^2 \#\left(\ln\frac{\mu^2}{s} + \frac{1}{\varepsilon} + c\right) + e_0^4 \#\left(\ln\left(\frac{\mu^2}{s}\right) + \frac{1}{\varepsilon} + c\right) + y_0^4 (\cdot\cdot)$$

$$Z^2 \lambda_r = \lambda_0 + (\#\lambda_0^2 + \#' e_0^4 + \#'' y_0^4)\left(\ln\frac{\mu^2}{s} + \frac{1}{\varepsilon} + \text{consts}\right)$$

"vertex" corrections <u>not</u> only thing involved in $\lambda_r - \lambda_0$ relation.

Renormalization

Problem is that $\left(\ln \frac{\mu^3}{Q^2} + \frac{1}{\varepsilon}\right)$ is
- Large, and
- dependent on (unphysical) UV regulation/cutoff.

Re-organize calculation to both
1) eliminate appearances of large $1/\varepsilon$ factors and
2) work in terms of physical stuff.

We saw that

$$Z\phi_r^2 = \phi_0^2 \quad \text{with } Z \text{ determined by self-energy}$$
$$Z^{-1} = (1 + p^2\text{-piece of self-energy } \Pi) \quad \frac{\partial \Pi}{\partial p^2}$$

$$Z^2\lambda_r = \lambda_0 + \text{1-loop vertex corrections}$$

Solve these relations for ϕ_0, λ_0 in terms of renormalized quant.

$$\phi_0 = \phi_r\sqrt{Z} = \phi_r\left(1 - \tfrac{1}{2}p^2\text{-piece of } \Pi\right)$$
$$\lambda_0 = \lambda_r\left(1 + (Z^2 - 1)\right) - \text{1-loop vertex corr.}$$

$$\phi_0 = \phi_r + \left(e^2, y^2 \text{ terms, with } \tfrac{1}{\varepsilon}\text{'s}\right) + \mathcal{O}(e^4, y^4) = \phi_r + \underbrace{(Z^{1/2} - 1)\phi_r}$$
$$\lambda_0 = \lambda_r + \left(\lambda e^2, \lambda^2 e^4 \text{ terms with } \tfrac{1}{\varepsilon}\text{'s}\right) + \mathcal{O}(e^6, y^4 e^2, \ldots)$$
$$= \lambda_r + \underbrace{(Z^2 - 1)\lambda - (\text{vertex})}$$
$$= \lambda_r + \underbrace{\delta\lambda_1}_{e^4} + \underbrace{\delta\lambda_2}_{e^6} + \ldots \text{ pieces...}$$

formally, 1 or more powers of (e^2, λ, y^2) higher.

Call these high-order terms "counterterms" & postpone to

Example: scattering again

$\langle \phi_r^4 \rangle_{\substack{\text{Amp.}\\ \text{conn.}}} =$ 

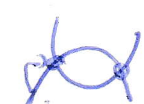
$$\lambda_r + \#e^4\left(\ln\frac{\mu^2}{s} + \frac{1}{\bar{\varepsilon}} + c\right) - \#e^4\left(\ln\frac{\mu^2}{s_0} + \frac{1}{\bar{\varepsilon}} + c\right)$$

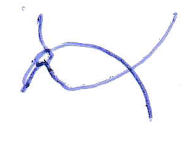
1-loop effect

1-loop counterterm from $\lambda_0 \neq \lambda_r$

Claim: the $\frac{1}{\bar{\varepsilon}}$ and $\ln(\mu^2)$ will cancel between these loop- and counterterm- contributions. Always.

Key: every divergence comes from few-leg, few-p-power subdiagram which can be reproduced by an $\mathcal{L}$ element/term

Interpretation: λ_r means  λ_0 + divergent piece of 

where, note, these vertices are λ_r's, as they are λ_0 + divergent piece of  ...

Implementation: when you encounter divergence, subtract it off, plus const term such that result obeys your renorm. scheme definition. EG,

scalar thy

 Σ propagator $= \frac{1}{\ldots}$

$\Pi(p) =$ (nontrivial func. of p^2)

$+ \underbrace{(A p^2 + B)}_{\text{counterterm}}$ w. A, B fixed chosen such that

~~tt~~ $\Pi(p^2 = M^2) = 0, \quad \frac{\partial}{\partial p^2} \Pi(p^2 = m^2) = 0$

Whatever $\frac{1}{\varepsilon}$ and $\ln(\mu^2)$ dependence is in $\Pi(p^2)$ nontrivial loop part,

$Ap^2 + B$ must contain opposite contrib. So $1/\varepsilon$, $\ln(\mu^2)$ disappear.

They do appear in relation, λ_r vs λ_0, Z vs 1

but that's OK.

This on-shell methodology is good enough if you have problem w. 1 physical scale and never ask about same thy. at disparate scales. EG, if you want IR QED.