# Renormalization Of QED



$$\mathcal{L}(A_0^\mu, \psi_0) = -\frac{1}{4} F_0^{\mu\nu} F_{\mu\nu 0} + \bar{\psi}_0 \left( i(\not\partial + i e_0 \not A_0) - m_0 \right) \psi_0$$

$F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$ is $F$ in terms of $A_0$.

Define $A_0 = Z_A^{1/2} A_r$, $\psi_0 = Z_\psi^{1/2} \psi_r$, $e_0 = \underbrace{(e_r - e_0)}_{\text{"}\delta e\text{"}} + e_0$

$$\mathcal{L} = -\frac{1}{4} Z_A F_r^{\mu\nu} F_{\mu\nu r} + Z_\psi \bar{\psi}_r \Big( i\not\partial - \underbrace{e_0 Z_\psi Z_A^{1/2}}_{\equiv Z e_r} \not A_r - m_0 \Big) \psi_r$$

$Z_\psi m_0 = m_r + \delta m$

<u>Rewrite</u>

$$\mathcal{L} = -\frac{1}{4} F_r^{\mu\nu} F_{\mu\nu r} + \bar{\psi}_r (i\not\partial - e_r \not A_r - m_r) \psi_r \quad \Big] \text{ Terms}$$

$$-\frac{1}{4} \underset{Z_A - 1}{\delta_A} F^2 + \bar{\psi}_r \left[ i \underset{Z_\psi - 1}{\delta_\psi} \not\partial - \underset{Z_\psi m_0 - m_r}{\delta m} \right] \psi_r - e \underset{\frac{e_0}{e} Z_\psi Z_A^{1/2} - 1}{\delta_e} \bar{\psi}_r \not A_r \psi_r \quad \Big] \text{ Counter-Terms}$$

Counter-terms inserted at loop level to absorb divergences & enforce renormalization conditions

Let's actually do it!

$\Sigma = \Sigma_1(p^2) + \not p \, \Sigma_\gamma(p^2)$

Conditions:

$$\langle \psi_r \bar{\psi}_r \rangle_{\text{loop}} = \frac{i}{\not p - m - \Sigma(p)}$$

need: $\Sigma(\not p = m) = 0$ right $m$

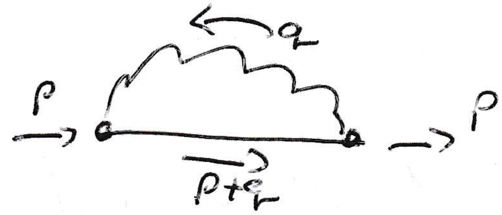
$\frac{d\Sigma_\gamma}{dp} = 0$ $Z$ correct

$$\langle A_r^\mu A_r^\nu \rangle_{\text{loop}} = \frac{-i g^{\mu\nu}}{p^2 - \Pi(p^2)}$$

need $\frac{d\Pi}{dp^2}(p^2 = 0) = 0$,

$\Gamma^\mu$ need $-ie\Gamma^\mu(p' - p = 0) = -ie\gamma^\mu$

Compute $\Sigma$




$$\underset{1\ \text{loop}}{\Sigma} = i(-e)^2 \int \frac{d^D q}{(2\pi)^D} \frac{-i g^{\mu\nu}}{q^2 + i\epsilon} \gamma_\mu \frac{i}{\not{q} + \not{p} - m + i\epsilon} \gamma_\nu$$

note, $e_r$, $m_r$, etc. not bare ones.
1-loop in Renorm. $\mathcal{L}$ no CT's.

Good practice: $\gamma_\mu \not{p} \gamma^\mu = ?$

$\gamma_\mu m \gamma^\mu = ?$

$$g^{\mu\nu} \gamma_\mu m \gamma_\nu = m g^{\mu\nu} \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = m g^{\mu\nu} g_{\mu\nu} = mD$$

not $4m$

$$p^\alpha g^{\mu\nu} \gamma_\mu \gamma_\alpha \gamma_\nu = p^\alpha g^{\mu\nu}\Big[-\gamma_\alpha \underbrace{\gamma_\mu \gamma_\nu}_{g_{\mu\nu}} + \underbrace{(\gamma_\alpha \gamma_\mu + \gamma_\mu \gamma_\alpha)}_{2 g_{\mu\alpha}} \gamma_\nu\Big]$$

$$= -D p^\alpha \gamma_\alpha + 2 p^\alpha g_{\mu\alpha} g^{\mu\nu} \gamma_\nu$$

$$= (2-D)\not{p} \quad \text{not } -2\not{p}$$

Also use Feynperson denominator trick ... we saw it.
Algebra $\rightarrow$

$$\Sigma(p) = i e^2 \int dx \int \frac{d^D q}{(2\pi)^D} \frac{Dm + (2-D)(\not{q} + (1-x)\not{p})}{(q^2 - x m^2 + x(1-x)p^2 + i\epsilon)^2}$$

Wick rotate $i\,dq^0 \rightarrow dq^0_E$, $q^2 \rightarrow -q^2$, $\not{q} \rightarrow \not{q}_E$

$$\Sigma(p) = \underset{\underset{\text{almost forgot.}}{\mu^{4-D}}}{\wedge} e^2 \int_0^1 dx \int \frac{d^D q_E}{(2\pi)^D} \left[ \frac{Dm + (2-D)((1-x)\not{p} + \not{q})}{(q_E^2 + \underbrace{x m^2 - x(1-x)p^2}_{M^2})^2} \right.$$

Angles: Denominator has no $q \cdot p$.

Only $q^2$ & $p^2$.

$$\int d^D q \frac{1}{(q^2+M^2)^2} \times \left[ \underset{\text{even}}{\underset{1}{q^2}} \quad \underset{\underline{\underline{\to 0}}}{\underset{\text{odd}}{\underset{1}{\not{q}}}} \quad \underset{\text{again}}{\underset{\text{even}}{\underset{=}{\not{p}\, q^0}}} \quad \underset{\underline{\underline{??}}}{q^\mu q^\nu} \right.$$

even in each $\vec{q}$

Not today, but maybe some day?

$$\int \frac{d^D q}{(2\pi)^D} \frac{q_\mu q_\nu}{(q^2+m^2)^\alpha}$$ Must be $\propto g_{\mu\nu}$ only available tensor structure.

Not $p_\mu p_\nu$ as no correlation between $\vec{q}, \vec{p}$ directions!

Call it $K g_{\mu\nu}$.

Apply $g^{\mu\nu}$: $g^{\mu\nu}(K g_{\mu\nu}) = K g^{\mu\nu} g_{\mu\nu} = DK$

$$g^{\mu\nu}\left[\int \frac{d^D q}{(2\pi)^4} \frac{q_\mu q_\nu}{(q^2+m^2)^\alpha}\right] = \int \frac{d^D q}{(2\pi)^4} \frac{q^2}{(q^2+M^2)^\alpha} = DK \quad \text{so,}$$

$$\int \frac{d^D q}{(2\pi)^D} \frac{q_\mu q_\nu}{(q^2+m^2)^\alpha} = \frac{g_{\mu\nu}}{D} \int \frac{d^D q}{(\pi)^D} \frac{q^2}{(q^2+M^2)^\alpha}$$ .... Handle All Tensors

Also recall : Dimensional Regularization 

Tricks

$$\int \frac{d^D q}{(2\pi)^D} \frac{(q^2)^A}{(q^2+M^2)^B} = \frac{(M^2)^{\frac{D}{2}+A-B}}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty \frac{(\bar{q}^2)^{\frac{D}{2}-1+A} \, d(\bar{q}^2)}{(\bar{q}^2+1)^B}$$

Angle integration, scaling out dimensions $\bar{q}^2 = q^2/M^2$

Call $y = \frac{\bar{q}^2}{\bar{q}^2+1}$ $\quad d\bar{q}^2 = \frac{dy}{(1-y)^2}$ , $\bar{q}^2 = \frac{y}{1-y}$ , $1+\bar{q}^2 = \frac{1}{1-y}$

$$\rightarrow \frac{(M^2)^{\frac{D}{2}+A-B}}{(4\pi)^{D/2}\Gamma(D/2)} \underbrace{\int_0^1 dy \, y^{\left(\frac{D}{2}+A-1\right)} (1-y)^{(B-A-D/2-1)}}_{\frac{\Gamma\left(\frac{D}{2}+A\right)\Gamma(B-D/2-A)}{\Gamma(B)}}$$

$\Gamma(B-D/2-A)$ : how divergent is it in the UV? $>0$ - not, $=0$ - log, $<0$ - power

$\Gamma(D/2+A)$ usually cancels $\Gamma(D/2)$ eg.

$$\frac{\Gamma(D/2+1)}{\Gamma(D/2)} = \frac{D}{2} \qquad \frac{\Gamma(D/2+2)}{\Gamma(D/2)} = \left(\frac{D}{2}\right)\left(\frac{D}{2}+1\right) \quad \ldots$$

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Return to $\Sigma$. Do Algebra

$$\Sigma(p) = \frac{e^2}{(4\pi)^{D/2}} \int_0^1 dx \, \Gamma(2-D/2) \left[\frac{xm^2 - x(1-x)p^2}{\mu^2}\right]^{1-D/2} \left(Dm + (2-D)(1-x)\not{p}\right)$$

Full 1-loop: $\underset{\not{p}-m}{\underline{\qquad\qquad}}$ +  $\Sigma(p)$ + -o- 1-loop Counterterm $(Z_\psi - 1)\not{p} - (Z_\psi m_0 - m_r)$

$\downarrow$

$$S^{-1} = \underbrace{\not{p}-m}_{\text{tree}} \underbrace{-\Sigma}_{\text{1 loop}} \underbrace{-(\delta\not{p} - \delta m)}_{\text{Counterterm}}$$

We need: $\frac{d}{d\not p}$ of $\gamma^\mu$ part of $\bar{S}^{-1}$ be $1$

Value of $\bar{S}^{-1}$ be $0$ ) at $p^2=m^2$, eg,

$(\not p+m)\,\bar{S}^{-1}=0$ mass

$$(\not p+m)\Sigma_{\cdot} = \delta m = \frac{e^2 m}{(4\pi)^{D/2}}\int_0^1 dx\, \frac{\Gamma(2-D/2)\,(D+(2-D)(1-x))}{\left(\frac{x^2 m^2}{\mu^2}\right)^{\frac{4-D}{2}}}$$

$$\frac{d}{d\not p}\Sigma(p) = (Z_4-1) = \frac{-e^2}{(4\pi)^{D/2}}\int_0^1 dx\, \frac{\Gamma(2-D/2)}{(x^2m^2/\mu^2)^{2-D/2}}\left(D(1-x)+\left(\frac{D}{2}-2\right)\frac{2x(1-x)m^2}{x^2m^2}\right)$$

$\cancel{\frac{2x(1-x)m^2}{x^2m^2}\left(4-2(1-x)+\left(\frac{D}{2}-2\right)(1-x)\right)}$

If we had used $\dfrac{g^{\mu\nu}+(\xi-1)\frac{q^\mu q^\nu}{q^2}}{q^2}$ and not $\dfrac{g^{\mu\nu}}{q^2}$ ...

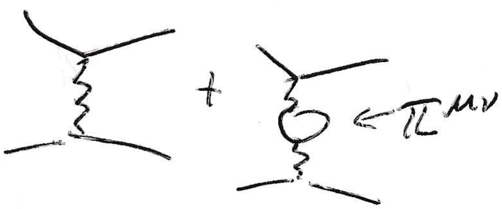
$\Sigma$ would have an added term $i\displaystyle\int\frac{d^Dq}{(2\pi)^D}\,\frac{4mq^2+2\not q\, q\cdot p-q^2\not p}{(q^2+i\varepsilon)((q+p)^2-m^2+i\varepsilon)}$

$Dm \to (D+\xi-1)m$ ...

Size of divergences not same. $Z_4$ not same!

Physical answers the same!

Self-Energy:




Last term $\Pi^{\mu\nu} = (q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2)$

$$\Pi(q^2) = \frac{-e^2}{(4\pi)^{D/2}} \int_0^1 dx \left[\frac{\mu^2}{m^2 - x(1-x)q^2}\right]^{2-D/2} \Gamma(2-D/2) \times 8x(1-x)$$

We need $q^2 + (Z_A - 1)q^2 - \Pi(q^2) = q^2$ as $q \to 0$

$$Z_A - 1 = -\frac{e^2}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)}{(m^2/\mu^2)^{2-D/2}} \underbrace{\int_0^1 dx\, 8x(1-x)}_{4/3} \quad \text{Done!}$$

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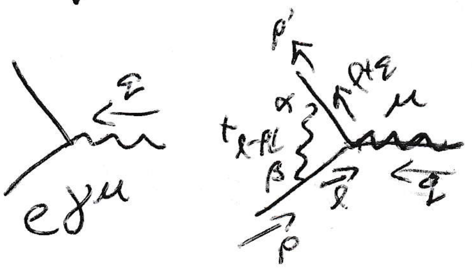
Vertex




$$e \times -ie^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{g^{\alpha\beta}}{(\ell-p)^2} \gamma_\alpha \frac{(\not\ell + \not q + m)}{(\ell+q)^2 - m^2} \gamma^\mu \frac{\not\ell + m}{\ell^2 - m^2} \gamma_\beta$$

rename $\ell - p \to \ell$

$\ell \to \ell + p$

at $q=0$, $$\gamma^\mu - ie^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{g^{\alpha\beta}}{(\ell-p)^2} \gamma_\alpha \frac{\not\ell + m}{\ell^2 - m^2} \gamma^\mu \frac{\not\ell + m}{\ell^2 - m^2} \gamma_\beta$$

Almost like integral for $\Sigma \to \dfrac{i}{\not\ell + \not p - m}$ in $\Sigma$ is now $\underbrace{\dfrac{\not\ell + \not p + m}{(p+\ell)^2 - m^2} \gamma^\mu \dfrac{\not\ell + \not p + m}{(p+\ell)^2 - m^2}}$

Call $\not p + \not\ell - m \equiv A$.

$$\frac{d\Sigma}{dp_\mu} = \frac{d(1/A)}{dp} = \frac{1}{A}\frac{dA}{dp}\frac{1}{A} = \text{exactly this}. \quad \frac{d\Sigma}{dp_\mu} = \Gamma^\mu(q \to 0)$$

(as A is a matrix) Counterterm exactly same

Counterterm for $\Sigma$: $\quad \bar{\psi}_0 \not{\partial} \psi_0 = Z_\psi \bar{\psi}_r \not{\partial} \psi_r$

CT: $Z_\psi - 1$

Counterterm for $\Gamma$: $\quad \bar{\psi}_0 e_0 A_0 \psi_0 = Z_\psi Z_A^{1/2} \frac{e_0}{e_r} \times \bar{\psi}_r e_r A_r \psi_r$

CT: $Z_\psi Z_A^{1/2} \frac{e_0}{e_r}$

CT's are same: $Z_\psi - 1 = \frac{e_0}{e} Z_\psi Z_A^{1/2} - 1$ $\underline{\text{or}}$ $\frac{e_0}{e_r} Z_A^{1/2} = 1$

$Z_A$ alone determines charge renorm. $\quad \boxed{e_r^2 = Z_A e_0^2}$

Expand expression for $Z_A$:

$$Z_A - 1 = \frac{-e^2}{16\pi^2} \frac{4}{3} \left(\frac{4\pi\mu^2}{m^2}\right)^{\overset{\varepsilon}{\overset{\shortparallel}{2-D/2}}} \Gamma(\overset{\varepsilon}{\overset{\shortparallel}{2-D/2}}) = \frac{-e^2}{12\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln(\mu^2/m^2)\right]$$

$\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \equiv \frac{1}{\varepsilon}$ since $-\gamma_E + \ln 4\pi$ $\underline{\text{always}}$ accompanies $1/\varepsilon$.

$\underline{\text{If}}$ I choose $\mu = m$ $\rightarrow$ subtract $\underline{\text{precisely}}$ $1/\varepsilon$ as Counter Term.
(Not true at 2+ loops)

For $\Sigma$:

$$\Sigma(p) = \frac{e^2}{16\pi^2} \left[(4m - \not{p}) \frac{1}{\varepsilon} - (4m - \not{p}) \underbrace{\int_0^1 dx \ln \frac{xm^2 - x(1-x)p^2}{\mu^2}} + \not{p} - 2m\right]$$

ugly, but for $p^2 = m^2 \Rightarrow \ln \frac{m^2}{\mu^2} - 2$.
But $\frac{\partial}{\partial p}$ $\underline{\text{not}}$ so simple!

Removing $\frac{1}{\varepsilon}$ is $\underline{\text{not}}$ enforcing on-shell conditions! $\underline{\text{Even}}$ $\mu = m$!
Differ by $\underline{\text{constants}}$

| On-shell Scheme | $\overline{MS}$ scheme |
| --- | --- |
| Set counterterms to remove both $1/\varepsilon$ and const's so on-shell cond. hold. | Counterterms are precisely $1/\varepsilon$ factors. Pick particular convenient $\mu$ |
| ↓ | ↓ |
| Good: $e_r^2 = e_{meas}^2$ correl func → $\mathcal{M}$ | Good - easy |
| Bad: CT complicated. Inflexible when energy scales high ~~E~~ $E \gg m_e$ | Bad - corrections between matrix elements & correl. Func, $m_r$ and $m$ and for $\mu \neq m$, $e_r^2$ and $e_{meas}^2$ |

$\overline{MS}$ - lets us learn how coupling $e^2$ depends on scale in nice elegant way.