Symmetries, Goldstone Theorem

Consider N-component scalar thy

$$\mathcal{L} = \frac{1}{2}\sum_n \partial_\mu \varphi_n \partial^\mu \varphi_n - \frac{\lambda}{4}\left(\sum_n \varphi_n^2\right)^2 + \frac{\mu^2}{2}\sum_n \varphi_n^2$$

note sign.

Nothing wrong with sign. Theory stable thanks to λ term.
But physics profoundly different!

Vac: state of lowest energy

$$\mathcal{H} = \frac{1}{2}\left(|\nabla\varphi_n|^2 + \dot{\varphi}_n^2\right) + \frac{\lambda}{4}(\varphi_n\varphi_n)^2 - \frac{\mu^2}{2}\varphi_n\varphi_n$$

summation convention $n = 1 \dots N$

$\varphi = 0$: energy 0

$\varphi = \frac{\mu}{\sqrt{\lambda}}$ ~~$\sqrt{\cdot}$~~ : energy $-\frac{\mu^4}{4\lambda}$ is lower! That's the vacuum.

But which <u>component</u> should have $\varphi \neq 0$?

Theory <u>had</u> a symmetry $\varphi_n \to \mathcal{O}_{nm}\varphi_m$ $\quad \mathcal{O}^T = \mathcal{O}^{-1}$

orthogonal $N \times N$ mat.

Dim. of group : $\frac{N(N-1)}{2}$ ~~such matrices independent~~

Choosing which φ gets nonzero "Vacuum Expectation Value" VEV breaks symmetry. I can <u>use</u> symm to rotate to state where φ_1 takes $+\mu/\sqrt{\lambda}$ value, others take 0.

(But any choice on sphere S^{N-1} of $|\varphi| = \mu/\sqrt{\lambda}$ in any direc. is just as good — call this N-1-sphere the Vacuum Manifold)

Rewrite $\varphi_1 = \sigma + \mu/\sqrt{\lambda}$ $\quad \sigma$ - fluct. $\mu/\sqrt{\lambda} \equiv v$ = fixed VEV.

Rewrite $\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\sum^N \partial_\mu\varphi_n\partial^\mu\varphi_n + \cancel{\dots}$ potential

Potential: $\frac{\mu^2}{2}\varphi_n^2 = \frac{\mu^2}{2}(\sigma+v)^2 + \frac{\mu^2}{2}\varphi_n^2$ ($n=2..N$)

$$-\frac{\lambda}{4}\left(\Sigma\varphi_n^2\right)^2 = -\frac{\lambda}{4}\left(\sigma^2+2v\sigma+v^2+\Sigma\varphi_n^2\right)^2$$

v-only terms $\frac{\mu^2}{2}v^2 - \frac{\lambda}{4}v^4$ but $v^2=\frac{\mu^2}{\lambda}$ $\Rightarrow \frac{\mu^2 v^2}{4}$ negative of vac energy

linear in σ terms (lin in fields) $\mu^2\sigma v - \lambda\sigma v^3 = 0$ no lin term (how we chose v!!)

quad. in fields: $\underbrace{\frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{2}\sigma^2 v^2 - \lambda\sigma^2 v^2}_{-\lambda v^2\sigma^2} \; \underbrace{+\frac{\mu^2}{2}\varphi_n^2 - \frac{\lambda}{2}v^2\varphi_n^2}_{\text{cancel: } \lambda v^2=\mu^2}$

$= -\mu^2\sigma^2$

$= -\frac{(2\mu^2)}{2}\sigma^2$

$\boxed{\text{mass}^2 = 2\mu^2}$

$0\,\varphi_n^2$ these fields are all massless!!

Cubic in fields $-\lambda v\sigma^3 - \lambda v\sigma\varphi_n^2$ they now have cubic int's. All $\propto\sigma$.

Quartic $-\frac{\lambda}{4}\left(\sigma^4 + 2\sigma^2\varphi_n^2 + \sum_{nm}\varphi_n^2\varphi_m^2\right)$ same as before.

$= (\sigma^2+\varphi_n^2)^2$

Looks like thy of 1 massive, $N-1$ massless scalars, $O(N-1)$ symmetric.

Note: original $\frac{N(N-1)}{2}$ symm. gen.

Now, $\frac{(N-1)(N-2)}{2} = \frac{(N)(N-1)}{2} - (N-1)$ symm gen

lost symm gen = # new massless scalars. No accident. Theorem

Step back & think of symmetry

Fields φ_n $n=1\ldots N$

Transform under $\varphi_n \to \mathcal{O}_{nm}\varphi_m$ $\quad \mathcal{O}^{-1}_{nm} = \mathcal{O}^T_{mn}$

so $\varphi_n\varphi_n \to \underbrace{\mathcal{O}_{nm}\mathcal{O}_{np}}\varphi_m\varphi_p$

$\mathcal{O}^T_{mn}\mathcal{O}_{np} = (\mathcal{O}^{-1}\mathcal{O})_{mp} = \delta_{mp} \Rightarrow \varphi_m\varphi_m$ ✓

Group $\mathcal{O}(N)$. Connected part $SO(N)$ (missing $\varphi_1 \to -\varphi_1$
(special orthogonal) all others fixed)

$SO(N)$ element $\mathcal{O}_{mn} = \exp i\Theta_{mn}$ with Θ_{mn} pure imaginary and antisymm $\Theta_{nm} = -\Theta_{mn}$

$= \exp i\Theta^A T^A_{mn} \to$ basis of $\frac{N(N-1)}{2}$ imaginary antisymm.

eg, $\begin{bmatrix} 0 & -i & \\ i & 0 & \\ & & \ddots \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & -i & \\ 0 & 0 & 0 & \cdots \\ i & 0 & 0 & \\ & \vdots & & \end{bmatrix}$ etc.

Associated (Nöther) currents

$$j^\mu_A = \sum_{mn} \varphi_m \partial^\mu \varphi_n T^A_{mn} \quad (\text{eg, } i\varphi_1\partial^\mu\varphi_2 - i\varphi_2\partial^\mu\varphi_1 \text{ etc..})$$

are conserved $\partial_\mu j^\mu = 0$ in sense discussed last year

Symm. trans. carried out on states via unitary operators $\mathcal{U}(\mathcal{O})$

$\mathcal{U}(\mathcal{O})|\psi\rangle = |\text{"}\mathcal{O}\psi\text{"}\rangle$ symm transform of state $|\psi\rangle$

$\mathcal{U}(\mathcal{O}) = \exp i(\Theta^A Q^A)$ with $[Q^A, Q^B]$ same commut. rel. as $[T^A, T^B]$ obey.

Namely $Q^A = \int d^3x\, j^0_A$ charge in the A-current

($P^\mu = \int d^3x\, T^{\mu 0}$ integral of 4-mom. density, with $T^{\mu\nu}$ the

Symmetry means $\partial_\mu j^\mu = 0$

$\partial_0 j^0 = i[H, j^0]$. So $i[H, \int d^3x\, j^0_A]$

$= i\int d^3x\, \partial_0 j^0_A = i\int d^3x\, \vec{\partial}\cdot\vec{j} = 0$ (int. by parts)

~~Norma~~ Therefore $\mathcal{U}(\theta) H = H\,\mathcal{U}(\theta)$

$\underbrace{H\,\mathcal{U}(\theta)|0\rangle}_{\text{state has same energy as vacuum.}} = \mathcal{U}(\theta) H|0\rangle = E_{vac}\,\mathcal{U}(\theta)|0\rangle$

Either 1) $\mathcal{U}(\theta)|0\rangle = |0\rangle$ vacuum is invariant under symm.

means, $(1 + i\theta^A Q^A)|0\rangle = 1|0\rangle$

$Q^A|0\rangle = 0$

chg. kills vac state

States in rep's of symmetry: if $a|0\rangle = |a\rangle$ state

Then $\underbrace{\mathcal{U} a \mathcal{U}^{-1}}|0\rangle = \mathcal{U}|a\rangle$ rotation of state $|a\rangle$

not same if this not same as a: States $|a\rangle$ in same Rep's as operators $\hat{a}$.

Or 2) $\mathcal{U}(\theta)|0\rangle \neq |0\rangle$ but some other state!

Vacuum not invariant under symm. Huh?

eg, $Q^A|0\rangle \neq 0$ but is some new state. What is this state?

State where I rotate $\varphi_1 = v$ in some direction to some ~~to some~~ equivalent but different value.

Well if $j^\mu_A(x)|0\rangle \neq 0$ it must be something: call it $|x, A\rangle$

and that must be expressible in p-basis states $|p, A\rangle = \int d^3x\, e^{-ip\cdot x}|x, A\rangle$

$$\langle p, A| j^\mu_A(x)|0\rangle = f^\mu_A(p)e^{ip\cdot x} = \underbrace{p^\mu f_A(p^2)}_{}e^{ip\cdot x}$$

some properly norm.

what else could appear here? only 4-momentum is p^μ.

$\cancel{\partial_\mu}\ \langle p,A|\,\partial_\mu j^\mu_A(x)|0\rangle = 0 = \langle p,A|\underbrace{\partial_\mu e^{i\hat{p}\cdot x}}_{i\hat{p}_\mu}\, j^\mu_A(0)\, \underbrace{e^{-i\hat{p}\cdot x}}_{\mathbb{1}}|0\rangle$

$= i p_\mu \langle p,A|\, j^\mu_A |0\rangle$

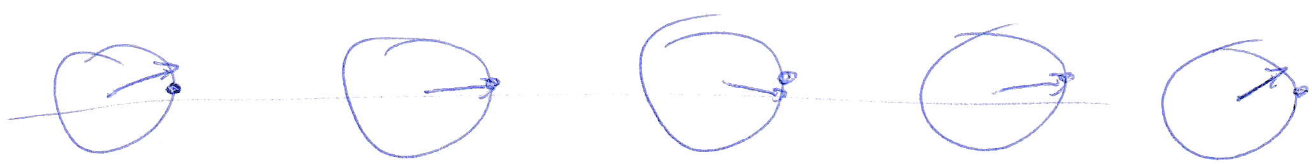
$0 = \underbrace{p_\mu p^\mu}_{p^2} f(p^2) e^{ip\cdot x}$ or $p^2 = 0$

Mode/state created by $j^\mu_A|0\rangle$ must be massless.

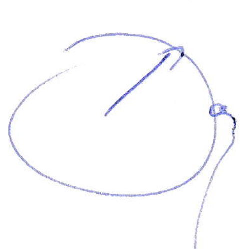
Physically, if I pick some $\vec{p}$ and apply

$$\left(\int e^{i\vec{p}\cdot\vec{x}}\, j^0(x)\, d^3x\right)|0\rangle$$

I get state, normalization $f(p^2)$, in which location on vac. manifold has been made to wiggle periodically



If wiggle arbitrarily long-wavelength,



uniform region w. displaced choice of vac. But that's just as good as original vac! No local way to know rest of Universe chose this value. So any energy costs only through gradients, that is, $\propto K$. Hence massless.

Scalar: characterized by displacement in field-space - unchg'd under space rotation.

Other q-#'s: same as operator Q_A.

Renormalization

General form of $\mathcal{L}$, shifed, is

$$\mathcal{L} = -\frac{\lambda_1}{4}\sigma^4 \mp \frac{\lambda_2}{2}\sigma^2\varphi_n^2 \mp \frac{\lambda_3}{4}(\varphi_n^2)^2$$

$$- g_1\sigma^3 - g_2\sigma\varphi_n^2$$

$$-\frac{\mu_1^2}{2}\sigma^2 - \frac{\mu_2^2}{2}\varphi_n^2$$

$$+\frac{(\partial\sigma)^2}{2} + \frac{(\partial\varphi_n)^2}{2}$$

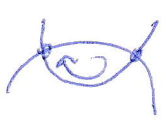
$\varphi_n^2 \equiv \sum_{n=2}^{N}\varphi_n^2$

without 1-comp

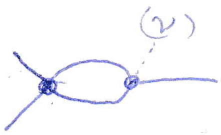
naively $\lambda_1 = \lambda_2 = \lambda$
$g_1 = \lambda v = g_2$
$\mu_1^2 = 2\mu^2$, $\mu_2^2 = 0$

How do I know Renormalization won't cause $\lambda_1 \neq \lambda_2 \neq \lambda_3$ or $\mu_2^2 \neq 0$??

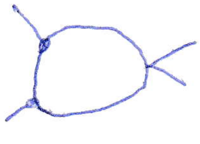
Because UV physics doesn't care about VEV :



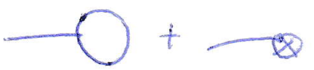
$\int \frac{d^D l}{(2\pi)^D}$ l huge... v finite, effect suppressed by at least 1 power of v/l. 4-pt: finite effects

3-pt 

3 legs but one power of g (or v)
Corrections only logarithmic.
Not independent either - determined by λ, μ^2 corrections

new diagrams, 

different for φ, σ ext lines,
are safe because of g^2 (or v^2) $\rightarrow \int \frac{d^D l \; v^2}{(l^2)^3}\ldots$

Similarly, when we choose v so  $=0$
so it stays the stable vac value, we find μ_2^2 remains 0 at loop level.