L6P1

Classical Background, Effective Action, $1PI$ generator

Recall $W = i \ln Z$

Suppose I take seriously current J as external source.

$$\frac{\delta W(J)}{\delta J(x)} = \frac{+i\delta}{\delta J(x)} \ln \int \mathcal{D}(\ldots) e^{i\int (\mathcal{L} + J\phi) d^D x}$$

$$= -\frac{1}{Z} \int \mathcal{D}(\ldots) \phi e^{iS \ldots} \quad = -\langle \phi \rangle_J$$

$\left(= -{}_J\langle 0 | \phi(x) | 0 \rangle_J \right.$ Vacuum in presence of $\left. J \right)$

Call this ϕ_{cl}.

$\phi_{cl} = \phi_{cl}[J]$ formally, function of J

Invertible, in principle: $J = J(\phi)$

Recall, $P = \frac{\delta}{\delta \mathring{Q}} \mathcal{L}$ $\quad P = P[\mathring{Q}]$ and $\mathring{Q} = \mathring{Q}[P]$

Was useful to define Legendre transform

$$H(P,Q) = P\mathring{Q} - \mathcal{L}(\mathring{Q}[P], Q)$$ solve $\mathring{Q}$ as func. of P to make formal func. of P, Q

Similarly I can define

$$\Gamma(\phi_{cl}) = -\int d^4y\, J(y)\, \phi_{cl}(y) - W(J[\phi_{cl}])$$

(solve J in terms of ϕ)

"Effective Action" or "Generator of 1PI diagrams" as we will see

$$\frac{\delta}{\delta \phi_{cl}(x)} \Gamma(\phi_{cl}) = -J(x) - \int_y \frac{\delta J(y)}{\delta \phi_{cl}(x)} \phi_{cl}(y) - \frac{\delta J(y)}{\delta \phi(x)} \underbrace{\frac{\partial}{\partial J(y)} W(J)}_{= -\phi_{cl}(y)} = -J(x)$$

Oh! they cancel!

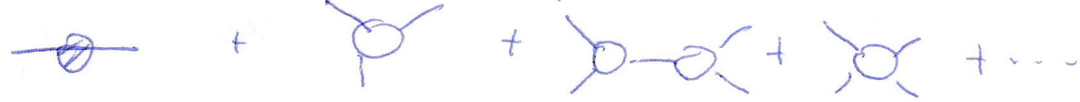
$$\frac{\delta}{\delta\phi_{cl}}\Gamma(\phi_{cl}) = -J \quad \text{and} \quad -\frac{\partial}{\partial J}W(J) = \phi_{cl}$$

Legendre transform inverted by another transform

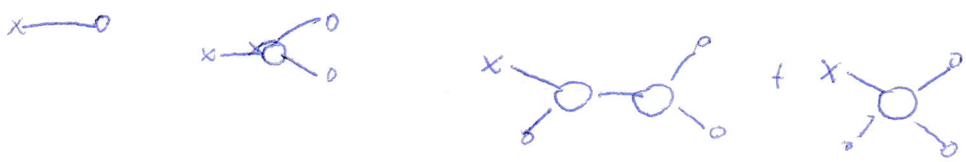
$$W = -\int J\phi - \Gamma(\phi[J]) \quad \text{Legendre transform pair}$$

Formal meaning: consider thy where ~~...~~ 3-pt function $\neq 0$

$$W = \frac{G_2}{2}JJ + \frac{G_3}{6}JJJ + \frac{G_4}{24}JJJJ + \dots$$



$$\phi = \frac{\delta W}{\delta J} = G_2 J + \frac{1}{2}G_3 JJ + \frac{G_4}{6}JJJ + \dots$$



What is J in terms of ϕ?

Lowest order: $J = G_2^{-1}\phi$

~~next order~~ $\cancel{J = G_2^{-1}\phi - \frac{1}{2}G_3 G_2^{-1}\phi G_2^{-1}\phi - \frac{1}{6}G_4(G_2^{-1}\phi)^3}$

If I mindlessly make this replacement, $J \to G_2^{-1}\phi$, on all high-order terms, I will get

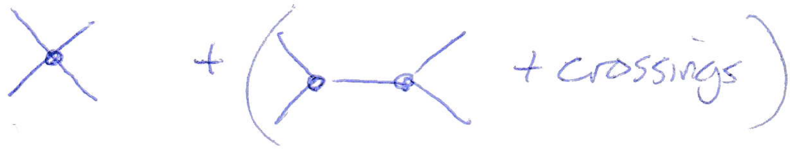
$$\Gamma(\phi_{cl}) = \underbrace{\phi G_2^{-1}\phi - \frac{1}{2}\phi G_2^{-1}G_2G_2^{-1}\phi}_{\frac{1}{2}(G_2^{-1})\phi\phi} - \frac{1}{6}(\phi G_2^{-1})^3 G_3 - \frac{1}{24}(\phi G_2^{-1})^4 G_4 + \dots$$

Amputated diagrams.

Remark: connected vs. 1-particle Irreducible

Consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{6}\phi^3 - \frac{\lambda}{24}\phi^4$ theory.

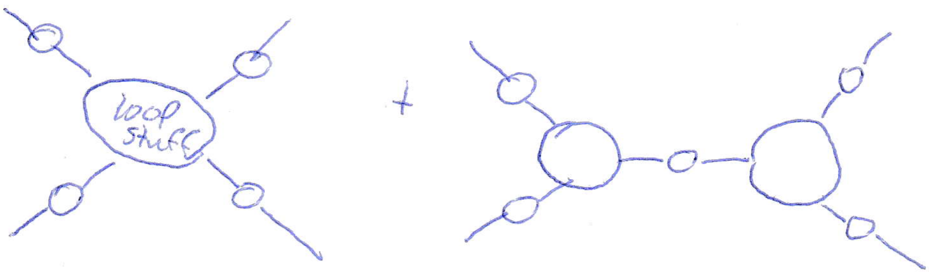
Lowest-order 4-point function =



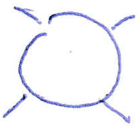
But if I know 3-pt function & propagator, I know

At loop level, I can write

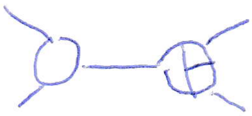
4-pt func. =



blobs: "any" loop stuff. Well, not quite any



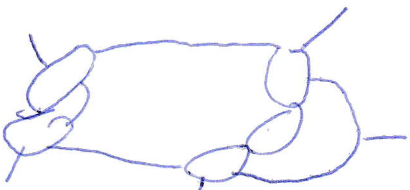
should not include



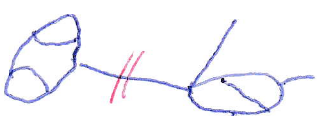
for instance as that's part of

To make distinction clearly, introduce

1-particle-Irreducible: an amputated N-pt ~~corr~~ diagram is said to be 1PI iff there is no single propagator you can cut to make it disconnected.



1PI 5-pt func

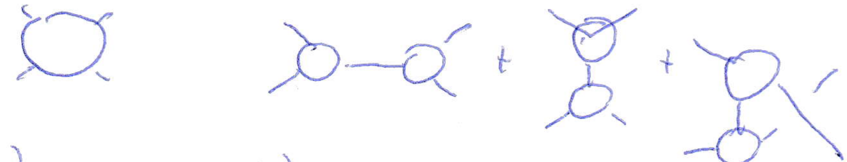


1-P-R diagram

¶ If I know the 1-part. Irreducibles, I can build everyone else.

$$G^{(3)}_{Amp} = G^{(3)}_{1PI} \quad \text{'cause it's amputated}$$

$$G^{(4)}_{Amp} = G^{(4)}_{1PI} + 3\, G^{(3)} G G^{(3)}$$



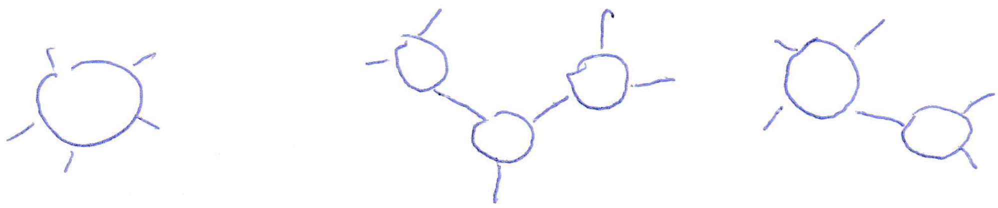
(stu)

(really, $G^{(4)}(xyzw) = G^{(4)}_{1PI}(xyzw) + \int d(ab)\, G^{3}(xya) G(ab) G^{3}(bzw)$

$+ \quad xz \quad yw$

$\quad\; xw \quad yz$

$$G^{(5)}_{amp} = G^{(5)}_{1PI} + G^{3} G G^{3} G G^{3} + G^{3} G G^{4} \text{ + permutations}$$



So $G^{(N)}_{amp}$ contains unneeded redundant info.

Claim: $\Gamma(\phi)$ = generating function of 1PI diagrams

Proof: Weinberg Vol II Ch. 16.1

What does it mean? Part 1

At lowest order

$$\Gamma(\phi_{cl}) = \frac{1}{2}\phi(p^2-m^2)\phi - \frac{g}{8}\phi^3 - \frac{\lambda}{24}\phi^4 \quad \text{same as Action!}$$

$$\text{or } \frac{1}{2}(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2)$$

whereas

$$W(J) = \frac{1}{2}J\frac{1}{p^2-m^2}J - J\frac{1}{p^2-m^2}g\frac{1}{p^2-m^2}J\,\frac{1}{p^2-m^2}J - (JG)^4\frac{\lambda}{24}$$

$$- \left((JG)^2 g\right)^2 G/8$$

looks nothing like action.

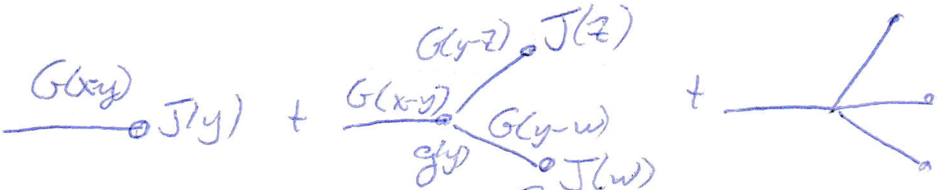
Classically, in background of Js, $\Box\phi = J + \frac{g\phi^2}{2} + \frac{\lambda\phi^3}{6}$

$$\phi(x) = \int G(x-y)J(y)\,d^4y \quad \text{direct influence}$$

$$+\int G(x-y)\frac{g}{2}\phi^2(y)\,dy + \int G(x-y)\frac{\lambda}{6}\phi^3(y)\,dy$$

because $\int G(x-y)\,dy$ is inverse of $\Box$ operator.

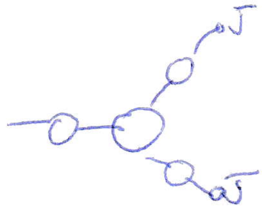
Inserting,

$\phi(x) =$ 

+ (each of those J's turned into $g\phi^2 + \lambda\phi^3$)

eg, trees  etc

At full Quantum level

$\langle \phi(x) \rangle = G_{full}(x-y) J(y) +$  $+ \ldots$

Use classical expressions but using full 1PI N-pt functions (not just 3&4!)

When $J=0$, $\langle \phi_{cl} \rangle$ need not $=0$, but $\dfrac{\delta \Gamma(\phi)}{\delta \phi_{cl}(x)} = J(x) = 0$

at $J=0$, Γ is an extremum of ϕ_{cl}.

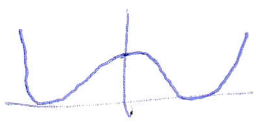
This allows computation of correct ϕ_{cl} value when explicit or spontaneous symm. breaking makes it $\neq 0$.

$V_{eff}(\phi) = \Gamma(\phi_{cl})$ evaluated for space-uniform ϕ_{cl}: "eff. Potential"

For $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_n \partial^\mu \varphi_n - \frac{\lambda}{4}(\varphi_n \varphi_n - v^2)^2$ theory,

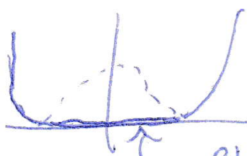
classically, $\underset{\text{4-volume}}{\Omega} \cdot \left(\frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 \ldots \right)$

at lowest perturbative order

$V_{eff}(\phi) =$  $=$ classical potential.

Loop-order:  modified

contains imaginary part

Nonperturbatively 

flat - $V_{eff}(\phi)$ must be convex

Why? $\langle \phi \rangle = 0$ if $\phi = \phi_1$ in half of Universe
$= \phi_2$ in other half $\}$ gives average of $V_{eff}(\phi_1), V_{eff}(\phi_2)$

"Maxwell construction"