Renormalization Group

$$-\frac{\lambda^2}{32\pi^2}\left(3\,\frac{1}{\varepsilon} + \ln\frac{\mu^2}{s} + \ln\frac{\mu^2}{t} + \ln\frac{\mu^2}{u} + \text{Const}\right)$$

$\frac{1}{\varepsilon}$: think of as $\ln\frac{\Lambda^2}{\mu^2}$

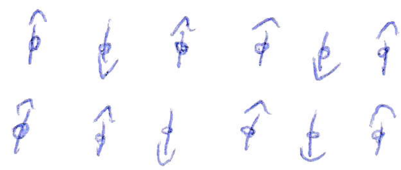
what's μ^2? How to think about this?

Wilson: imagine doing path $\int$ over UV keeping IR fixed.

Think of μ^2 as scale above which you $\int$ out.
Absorbing all $1/\varepsilon$ into renormalized couplings = integrating out scales down to μ.

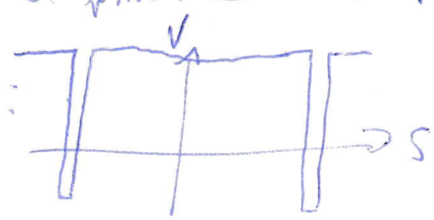
More explicit example: blocking

Consider Ferromagnetism near Fermi transition.



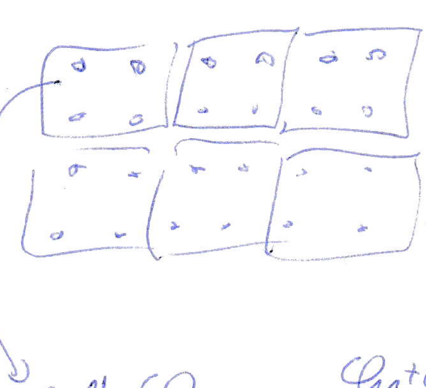
lattice of spins. Each is $\uparrow$ or $\downarrow$

Potential:



Far from polynomial

Define blocked spin: average over neighbors in block



Explicitly, path $\int$ is

$$\int d\varphi_{11}\, d\varphi_{12}\, d\varphi_{13} \cdots \quad d\varphi_{21}\, d\varphi_{22}\, d\varphi_{23} \cdots \qquad \text{Func(all } \varphi\text{'s)}$$

call $\varphi_{B11} = \frac{\varphi_{11}+\varphi_{12}+\varphi_{21}+\varphi_{22}}{4}$. Write as $\int d\varphi_{B11}\, d(\Delta\varphi_1)\, d(\Delta\varphi_2)\, d(\Delta\varphi_3)$

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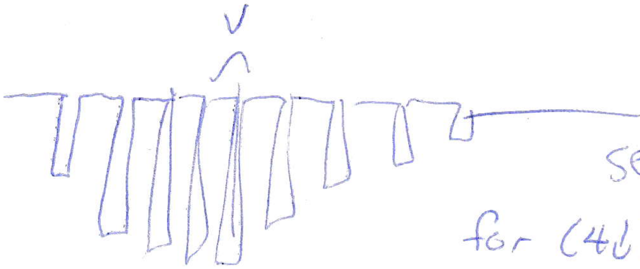
In principle I can arrange Path $\int$ as

$$\int d\varphi_{B11}\, d\varphi_{B12}\, d\varphi_{B13}\, d\varphi_{B21} \cdots \times \underbrace{\left(\int d^3\varphi_{\text{in}B11}\, d^3\varphi_{\text{in}B12} \cdots \; F(\text{all } \varphi\text{'s}) \right)}_{\text{"perform" to find } \tilde{F}(\varphi_B\text{'s})}$$

Very hard in practice but in principle possible.

What do we find?

1) potential is now

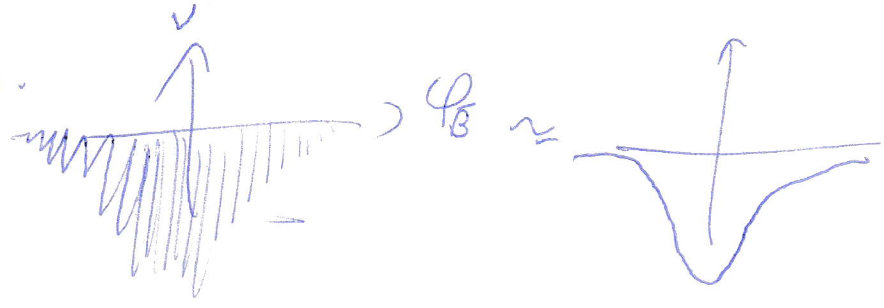


several minima for $(4\downarrow), (3\downarrow 1\uparrow), (2\downarrow 2\uparrow)$...

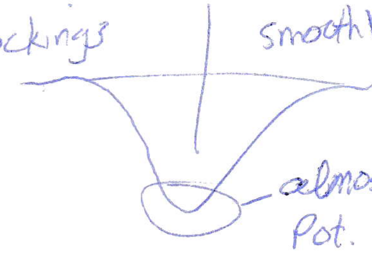
not all same depth! lower energy cost when spins align due to neighbor-interactions. But statistics favor $2\uparrow 2\downarrow$ over $4\downarrow$ (6 possibilities vs. 1 poss.)

Also block neighbor interac's not same as $N \times$ individual int's.

New block again.



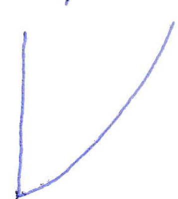
after a few blockings



almost 100% of path $\int$ has field down here.
Pot. is nicely polynomial

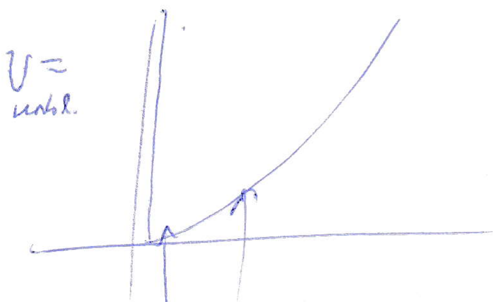
Why are there no non-analytic potentials in QFT?

Imagine I give you $V =$



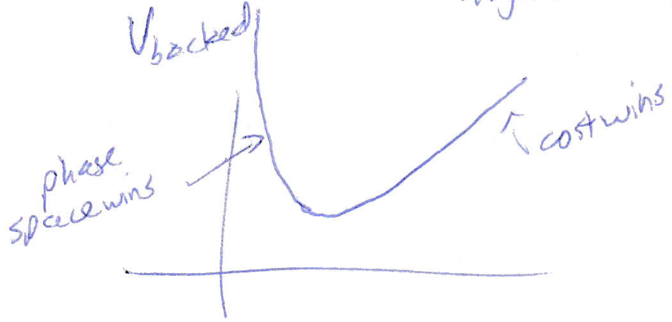
$\begin{cases} \varphi^2 & \varphi > 0 \\ \infty & \varphi < 0 \end{cases}$

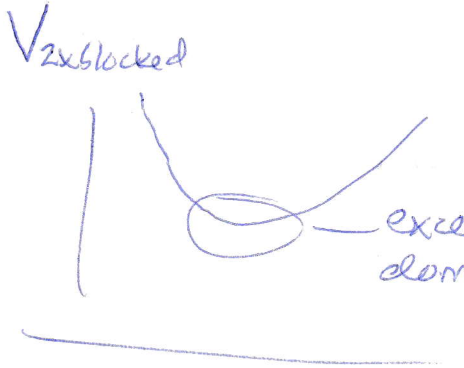
Block once



to land here, all spins must be in small neighborhood near zero as none can be negative.
small $\int$ region.

higher V for each φ, but wider $\int$ region





except ensemble dominated here.

Yes, large field deviations see funny V. But they almost never occur - captured by $(\varphi-\varphi_0)^6$ etc

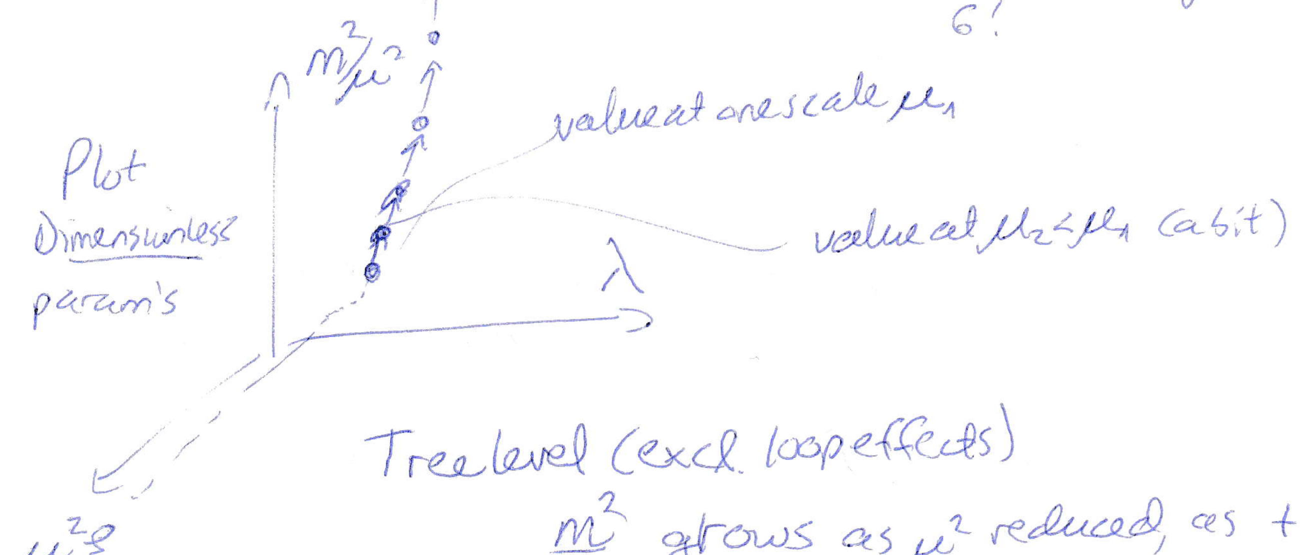
But we know why those are irrelevant already.

Philosophy 2: imagine integrating not in big steps but ~~the sm~~ smoothly in small steps (even continuously)

thy stays (generally) in same "space of theories" but parameters can change

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{\lambda}{24}\varphi^4 - \frac{m^2}{2}\varphi^2 - \frac{\xi}{6!}\varphi^6 \quad \text{say}$$

Plot Dimensionless param's



Tree level (excl. loop effects)

$\frac{m^2}{\mu^2}$ grows as μ^2 reduced, as +2 power

Gets more & more important in IR: RG Relevant

λ stays same as μ^2 reduced, 0-power

Importance stays same: RG Marginal

$\mu^2\xi$ shrinks, −2 power RG Irrelevant.

~~What about when μ reaches mass of some~~

Explicitly: loop corrections

Scatt process at fixed (s,t,u) must have definite μ-ind. value.

But m^2, λ etc. depend on μ.

removed by using $\lambda(\mu)$ not λ_0

$$\mathcal{M}(stu) = \lambda(\mu) - \frac{\lambda^2(\mu)}{32\pi^2}\left[\cancel{\frac{3}{\varepsilon}} + 3\ln\frac{\mu^2}{(stu)^{1/3}} + c\right]$$

$$\mu\frac{\partial}{\partial\mu}\mathcal{M} = 0 = \mu\frac{\partial\lambda}{\partial\mu} - \frac{\lambda^2}{32\pi^2}\times 6 - \frac{\lambda}{16\pi^2}\frac{\partial\lambda}{\partial\mu}\left(\ln\frac{\mu^2}{\sim}\right) + \dots$$

smaller than and only relevant at next order.

so $\mu\frac{\partial\lambda(\mu)}{\partial\mu} = 6\,\lambda^2 \ldots = \beta$

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Differential equation for μ-dep. of λ

Increase $\mu \to$ increase effective strength of coupling

Remark 1: I can now predict (part of) a 2-loop calc.

$$\mathcal{M} = \lambda - \frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\mu^2}{(\,)} + \text{const}\right) + \lambda^3\left(A \ln^2\frac{\mu^2}{(\,)} + B\ln\frac{\mu^2}{(\,)} + C\right)$$

$$0 = \mu\frac{\partial}{\partial\mu}\mathcal{M} = \beta - \frac{6\lambda^3}{32\pi^2} - \frac{6\lambda\beta}{32\pi^2}\ln\frac{\mu^2}{(\,)} + 4A\lambda^3\ln\frac{\mu^2}{(\,)} + 2B\lambda^3 + \dots$$

$\beta = \frac{3\lambda^2}{16\pi^2}$ at lowest order

Terms of order $\lambda^3\log\mu^2$ must cancel.

$$0 = -\frac{6\lambda}{32\pi^2}\cdot\frac{3\lambda^2}{16\pi^2}\ln\left(\frac{\mu^2}{(\,)}\right) + 4A\lambda^3\log\frac{\mu^2}{(\,)}$$

$A = \frac{9}{4(16\pi^2)^2}$ and I didn't need to do any additional calc.

I can't find "B" this way – it must be computed (along with const's next to $\lambda^2\ln(\,)$) to find λ^3-part of β.

Procedure finds $\lambda^2\ln \to \lambda^3\ln^2$, $\lambda^{n+1}\ln^n$ terms all at once.

Equivalent to solving $\mu\frac{\partial\lambda}{\partial\mu} = \frac{3}{16\pi^2}\lambda^2$

Let's run ahead, and come back next lecture to systematic exposition

call $t = \ln(\mu)$

$$\frac{d}{} = \frac{d}{} = \frac{d\mu}{}\frac{d}{} = \mu\frac{d}{d\mu} \cdot \quad \frac{d\lambda}{dt} = \frac{3}{16\pi^2}\lambda^2$$

$$\frac{d\lambda}{\lambda^2} = -d(1/\lambda) \qquad \frac{1}{\lambda} = \frac{-3}{16\pi^2}(t - \text{int. constant})$$

$$\frac{1}{\lambda} = \frac{3}{16\pi^2}\cancel{\ln\left(\frac{t_0}{t}\right)}(t_0 - t) = \frac{3}{16\pi^2}\ln\left(\frac{\mu_0^2}{\mu^2}\right)$$

μ_0: ultraviolet scale so $\mu_0^2/\mu^2 > 1$

As μ gets small, $\frac{1}{\lambda}$ gets big $\to$ λ gets small.

What about scale μ_0? What happens at/above?

"Landau Pole" (Pole because $\lambda \sim 1/(t_0 - t)$ is pole in λ in t-plane)

Landau - 1958 paper

Now λ^3 terms in β can change this - once λ large, do not believe calculation. But fancier studies - appears this really happens.

$\lambda\varphi^4$ thy w. finite coupling, as complete thy on all length scales, does not exist. To get $\lambda\varphi^4$ thy to exist, must have finite UV scale. As you go below this scale, $\lambda \to 0$. "Triviality problem"

Last remark: mass threshold

Consider QED of e^- and μ^- $\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_e(i\not{D} - m_e)\psi_e + \bar{\psi}_\mu(i\not{D} - m_\mu)\psi_\mu$

$M_\mu \simeq 200\, m_e$.

for scales $\mu > M_\mu$, thy of 2 charged particles.

Below $\mu = M_\mu$, though, only 1 particle can be produced!

Near $\mu = M_\mu$, use 2-part. thy as 

nontrivial p-dep.

Well below $\mu = M_\mu$, waste of time to have it in thy.