Teilchenphysik:

Lecture 10: Quantum Electrodynamics

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Our first genuinely useful theory: Quantum Electrodynamics

- Spin-$\frac{1}{2}$ particles: electrons
- Also protons or muons or ...
- Spin-1 particles: photons as A^μ field
- Interactions: $A_\mu J^\mu$

Feynman rules? Results for real quantities?
We'll see how far we get!

2: Feynman rules for Scalars

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Recall the Feynman rules for a scalar: $\mathcal{L} = \frac{1}{2}\phi(-\partial_\mu\partial^\mu + m^2)\phi + g\phi^3$

Object	Symbol	Value .
Incoming line		
Outgoing line		
Internal line (Propagator)		
Vertex		

3: Free spin-half

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$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Object	Symbol	Value
Incoming line (particle)		
Incoming line (antiparticle)		
Outgoing line (particle)		
Outgoing line (antiparticle)		
Internal line (Propagator)		

4: Spin 1 and vertex

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + eA_\mu J^\mu \simeq \frac{1}{2}A_\mu\partial_\nu\partial^\nu A^\mu - eA_\mu\bar{\psi}\gamma^\mu\psi$$

Object	Symbol	Value .
Incoming line		
Outgoing line		
Internal line (Propagator)		
Vertex		

5: What are these polarization things?

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As you know from classical EM, the A field is orthogonal to propagation and there are two polarization states:

$$\epsilon_\mu^{(1)},\ \epsilon_\mu^{(2)} \quad \text{with} \quad g^{\mu\nu}\epsilon_\mu^{(i)}(\epsilon_\nu^{(j)})^* = -\delta_{ij} \quad \text{and} \quad p^\mu \epsilon_\mu^{(i)} = 0$$

For instance, for p^μ in z direction and magnitude p, we have

$$p^\mu = \begin{bmatrix} p \\ 0 \\ 0 \\ p \end{bmatrix} \quad \epsilon_\mu^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \epsilon_\mu^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Plane polarizations.
Alternative, also useful: circular polarizations

$$\epsilon_\mu^+ = \frac{\epsilon_\mu^{(1)} + i\epsilon_\mu^{(2)}}{\sqrt{2}}, \qquad \epsilon_\mu^- = \frac{\epsilon_\mu^{(1)} - i\epsilon_\mu^{(2)}}{\sqrt{2}}$$

6: Example: electron scatters from a Muon

Vertices have same particle type going out as in!

7: Example: electron-electron scattering

Identical final states!

8: Wait – a minus sign?

The matrix element for $e^- e^- \to e^- e^-$ has two contributions.
The sign on one contribution isn't important since I will square.
Relative sign *is* important and there is a minus sign!

Why? Because e^- are fermions. The external state

$$|e_1 e_2\rangle = -|e_2 e_1\rangle$$

is antisymmetric on changing “labels” on the electrons.
The matrix element picks up a $-$ from this antisymmetry.

Also don't forget to integrate over half of momentum space, or use a factor $\frac{1}{2}$.

9: Example with photons: Compton

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10: How to evaluate these?

For *specific* in and out momenta and specific spins,
use the known forms of $u(p, s)$ etc and ϵ^i_μ.
Problems: final state $\bar{u}(p, s)$ change with changing p.
Most experiments don't measure spin / polarization state.
Most beams are an even mix of spins

More useful to ask about spin summed/averaged results

11: Spin sums and averages

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Average over initial spin states:
Usually you know the total number of particles in beam, not their spins.
Half are one spin, half are the other. *average.*

Sum over final spin states:
Your detector will detect the particle regardless of its spin.
Therefore you sould sum each possibility.

$$\left(\frac{1}{2}\right)^{\text{\#incoming}} \sum_{s_{\mathrm{in}}} \sum_{s_{\mathrm{out}}}$$

12: New simplifications

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I really want $\mathcal{M}^*\mathcal{M}$, not $\mathcal{M}$: if $\mathcal{M}$ contains $\bar{u}(p_4, s_4)\gamma^\mu u(p_1, s_1)$, I want

$$\bar{u}(p_4, s_4)\gamma^\mu u(p_1, s_1)\times(\bar{u}(p_4, s_4)\gamma^\nu u(p_1, s_1))^* = \bar{u}(p_4, s_4)\gamma^\mu u(p_1, s_1)\bar{u}(p_1, s_1)\gamma^\nu u(p_4, s_4)$$

(not obvious, you need $(\gamma^\nu)^*\gamma^0 = \gamma^0\gamma^\nu$)

Spin sum then includes terms like

$$\sum_s u(p_1, s)\bar{u}(p_1, s) = \gamma_\mu p_1^\mu + mc \qquad \text{and} \qquad \sum_s v(p_1, s)\bar{v}(p_1, s) = \gamma_\mu p_1^\mu - mc$$

(Prove in rest frame, use covariance...)
This is reason for weird normalization of u, v....

13: Turning things into traces

$$\text{Consider} \quad \bar{u}_1 \gamma^\mu u_2 \bar{u}_2 \gamma^\nu u_1$$

This is (row)(matrix)(column)(row)(matrix)(column)

$$\bar{u}_{1,i} \gamma^\mu_{ij} u_{2,j} \bar{u}_{2,k} \gamma^\nu_{kl} u_{1,l}$$

With indices explicit, I can put symbols where I want:

$$u_{1,l} \bar{u}_{1,i} \gamma^\mu_{ij} u_{2,j} \bar{u}_{2,k} \gamma^\nu_{kl}$$

Now interpret $u_l \bar{u}_i$ as a matrix with (l, i) indices, and $u_j \bar{u}_k$ as a matrix with (j, k) indices. l at front and back – trace of product of matrices

$$\mathrm{Tr}(u_1 \bar{u}_1)\gamma^\mu (u_2 \bar{u}_2)\gamma^\nu \quad \text{and} \quad \sum_{s_1 s_2} \mathrm{Tr}(u_1 \bar{u}_1)\gamma^\mu (u_2 \bar{u}_2)\gamma^\nu = \mathrm{Tr}(\gamma^\alpha p_\alpha + mc)\gamma^\mu (\gamma^\beta p_\beta + mc)\gamma^\nu$$

14: Dirac's notation

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Dirac thought it takes too long to write $\gamma^\mu p_\mu$, so he introduced a shorthand:

$$\gamma^\mu p_\mu \equiv \not{p}$$

Stop for a moment to appreciate the compactness of this expression. p^μ is a 4-component vector. γ^μ are four different 4×4 matrices. The tidy expression $\not{p}$ means:

$$\not{p} = \gamma^0 p_0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 = \begin{bmatrix} p^0 & 0 & p^z & p^x - ip^y \\ 0 & p^0 & p^x + ip^y & -p^z \\ -p^z & -p^x + ip^y & -p^0 & 0 \\ -p^x - ip^y & p^z & 0 & -p^0 \end{bmatrix}$$

Good notation plays a big role in making hard calculations feasible.

15: Calculating traces

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$$
\begin{aligned}
\text{start with } g^{\mu\nu} g_{\mu\nu} &= 4 \\
\text{and Tr } \mathbf{1} &= 4 \\
\text{and } \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= 2 g^{\mu\nu} \mathbf{1} \\
\text{Derive: Tr } \gamma^\mu \gamma^\nu &= 4 g^{\mu\nu} \\
\text{Tr } \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta &= 4(g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\beta}) \\
\text{Tr } \not{A} \not{B} &= 4 A \cdot B \\
\text{Tr } \not{A} \not{B} \not{C} \not{D} &= 4(A \cdot B\, C \cdot D + A \cdot D\, B \cdot C - A \cdot C\, B \cdot D) \\
\text{Tr (odd \# gammas)} &= 0 \\
\gamma^\mu \gamma_\mu &= 4 \\
\gamma^\mu \not{A} \gamma_\mu &= -2 \not{A} \\
\gamma^\mu \not{A} \not{B} \gamma_\mu &= 4 A \cdot B \\
\gamma^\mu \not{A} \not{B} \not{C} \gamma_\mu &= -2 \not{C} \not{B} \not{A}
\end{aligned}
$$

16: Real calculations

These tools let you do real calculations.
The amount of *algebra* involved is large.
But they are totally do-able, with some effort.
As long as we work to the lowest order.

Graph theoretically, “tree level.” Life gets tough when we do loops....

17: Summary

Feynman rules with spin are more complicated

- Incoming and outgoing lines have u, $\bar{v}$ and $\bar{u}$, v or $\epsilon_\mu, \epsilon^*_\mu$
- Vertices have γ^μ to connect spin, 4-vector indices
- Propagators have $\not{p} + m$ or $-g^{\mu\nu}$ numerators
- Identical external states lead to minus signs now

Calculations possible but clumsy in general. Nicer if you average/sum incoming/outgoing spins, corresponding to usual expt. situation
Reduces to evaluating traces of products of gamma matrices. Trace rules....