Teilchenphysik:

Lecture 12: QED to QCD

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Now we "know" QED. Complete theory for CM energies up to $E \simeq 2m_\pi = 260\text{MeV}$.
Go up in energy – encounter new bound states, made of quarks!

Two approaches:

- Historical: describe hadrons, see "pattern," intuit quarks
- Synthetic: describe answer – quarks and gluons – see what it predicts and how known hadrons come out.

I will follow "synthetic approach."

First, describe what happens at 10 to 50 GeV energy, where "almost free" quarks+gluons occur
Use this to define and describe the theory of strong interactions
Understand that and go down in energy to figure out how bound states form

Today: high-energy scattering and the constituents of QCD

2: Electron-Positron Annihilation

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Consider an electron-positron collider.

Scattering Annihilation .

Scattering $\to e^- e^+$, but annihilation $\to$ *any possible* charged particle, including new, unknown ones. Let's us explore and enumerate all charged particles!

3: Charged quark production

At energies $E > 10$ GeV, besides e^-, μ^-, τ^-, there are 15 new spin-$\frac{1}{2}$ charged particle types (and their antiparticles) we can produce:

- Three new particles with charge +2$/$3 and mass of few MeV
- Three new particles with charge $-1/3$ and mass of few MeV
- Three new particles with charge $-1/3$ and mass $\simeq$ 150 MeV
- Three new particles, charge +2$/$3 and mass $\simeq$ 1.4 GeV
- Three new particles, charge $-1/3$ and mass $\simeq$ 4.3 GeV

Each triplet of particles have *exactly* the same mass!

We call each *grouping of 3* particles a **flavor** (*udscb*)
and we distinguish the three equal-mass particles with a new label, called **color** because there are three (RGB) and for reasons we see soon.
So the 3 light, equal-mass, $Q = 2/3$ states are the **red, green, and blue up quarks**

4: Total annihilation cross-section

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Total cross-section to produce a spin-$\frac{1}{2}$ particle pair is

$$\sigma_{\text{annih}} = \frac{4\pi\, Q^2\, \alpha^2}{3s}$$

Ratio of *red up quark* production to μ production is $Q^2 = (2/3)^2 = 4/9$
Sum over the three *colors* of up quarks, get $4/3$.

Quarks fly on, interact, turn into who knows what.
But that doesn't affect *whether they got made.*

5: Hadron-to-muon ratio R

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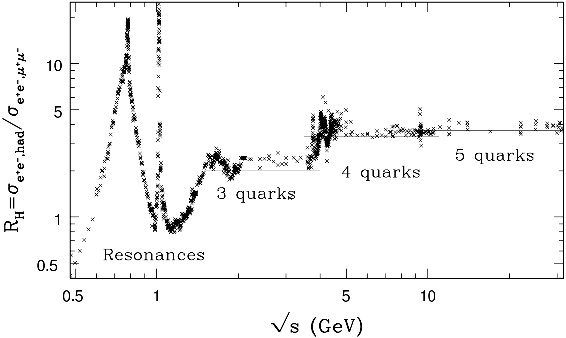
By *measuring* ratio of hadrons to muons, we can *count* the sum of particle-type times Q^2:



$$R \equiv \frac{\sigma_{\text{had}}}{\sigma_{\mu^-\mu^+}} = \sum_q Q_q^2 N_c$$

Allows us to "count" the quarks and see $Q^2 \times N_c$.

6: Isospin vs Color symmetry

Isospin is the *approximate* symmetry that the three lightest, $Q = 2/3$ particles interact *almost* like the three next-lightest, $Q = -1/3$ particles.

Color symmetry is the *exact* symmetry that the three “colors” of up quark are identical.
Therefore there are *symmetry transformations* rotating between them.
Put the three up-quark colors into a 3-component column vector:

$$u = \begin{bmatrix} u_r \\ u_g \\ u_b \end{bmatrix} \quad \text{symmetry:} \quad \begin{bmatrix} u_r \\ u_g \\ u_b \end{bmatrix} \rightarrow \begin{bmatrix} M_{rr} & M_{rg} & M_{rb} \\ M_{gr} & M_{gg} & M_{gb} \\ M_{br} & M_{bg} & M_{bb} \end{bmatrix} \begin{bmatrix} u_r \\ u_g \\ u_b \end{bmatrix}$$

As long as matrix M doesn’t change *length* of u column, this is a symmetry.

7: The group $U(3)$

What are these matrices M, transforming $u \to Mu$?

$$u^\dagger u = \begin{bmatrix} u_r^* & u_g^* & u_b^* \end{bmatrix} \begin{bmatrix} u_r \\ u_g \\ u_b \end{bmatrix}$$

Length squared of vector u

$$\to \begin{bmatrix} u_r^* & u_g^* & u_b^* \end{bmatrix} \begin{bmatrix} M_{rr}^* & M_{gr}^* & M_{br}^* \\ M_{rg}^* & M_{gg}^* & M_{bg}^* \\ M_{rb}^* & M_{gb}^* & M_{bb}^* \end{bmatrix} \begin{bmatrix} M_{rr} & M_{rg} & M_{rb} \\ M_{gr} & M_{gg} & M_{gb} \\ M_{br} & M_{bg} & M_{bb} \end{bmatrix} \begin{bmatrix} u_r \\ u_g \\ u_b \end{bmatrix}$$

$$= \begin{bmatrix} u_r^* & u_g^* & u_b^* \end{bmatrix} M^\dagger M \begin{bmatrix} u_r \\ u_g \\ u_b \end{bmatrix}$$

Dagger is transpose and conjugate

Unchanged if $M^\dagger M = \mathbf{1}$, that is, $M^\dagger = M^{-1}$. Called 3×3 unitary matrices $U(3)$

8: Generators of the Group: rotations

What is the most general such matrix?
This is a Lie group. I can “build” any element as a product of many many *nearly-identity* elements, just like with rotations, eg,

$$R(\theta_z) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \lim_{N\to\infty} \left(\begin{bmatrix} 1 & -\theta/N & 0 \\ \theta/N & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^N$$

With rotations, that matrix in the middle is

$$R = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\theta}{N} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^N = \exp\left(\theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

That was a rotation about the z axis. The most general possible rotation is:

$$
\begin{aligned}
R &= \exp\left(\begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix}\right) \\
&= \exp\left(\theta_z \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \theta_y \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \theta_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right)
\end{aligned}
$$

Why? Real unitary (orthogonal) matrix is exponential of a real *antiHermitian* (therefore antisymmetric) matrix.
Linear combination of three independent matrices with three coefficients.

If you like Hermitian matrices, you can put $-i$ in front of θ and i into each matrix.
Now matrices are *Hermitian*

10: Generators of $U(3)$

Now let's do the same thing with $U(3)$. Suppose M is almost the identity:

$$M = \mathbf{1} + iH$$

with H very small. Then $M^\dagger = \mathbf{1} - iH^\dagger$ and

$$M^\dagger M = (\mathbf{1} - iH^\dagger)(\mathbf{1} + iH) = \mathbf{1} + i(H - H^\dagger) + \mathcal{O}(H^2)$$

To linear order (enough for when we exponentiate) we need

$$H - H^\dagger = 0 \quad \text{or} \quad H = H^\dagger \quad \text{or } H \text{ is Hermitian.}$$

Hermitian: real components symmetric, imaginary components antisymmetric:

$$H = \begin{bmatrix} r_{11} & r_{12} + ig_{12} & r_{13} + ig_{13} \\ r_{12} - ig_{12} & r_{22} & r_{23} + ig_{23} \\ r_{13} - ig_{13} & r_{23} - ig_{23} & r_{33} \end{bmatrix}$$

Nine independent coefficients: six r_{ij} and three g_{ij}

11: A standard basis: Gell-Mann matrices

Standard basis: Gell-Mann matrices

$$H = \sum_{a=1}^{9} c_i \lambda_i \,,$$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \qquad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \lambda_9 = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Funny normalization? Chosen so $\mathrm{Tr}\, \lambda_a^2 = 2$ for each a. All Hermitian, all but λ_9 traceless.

12: But $U(3) = SU(3) \times U(1)$

The matrix λ_9, proportional to identity, commutes with all others. Its exponential is a pure phase $\exp(i\theta\lambda_9) = e^{i\theta\sqrt{2/3}}\mathbf{1}$.
Any matrix M can be written as a pure phase times a *unitary, determinant-1* matrix:

$$U(3) = SU(3) \times U(1)$$

The symmetry is really two symmetries: overall phases, and true rotations of the 3-color basis.

The way the theory behaves under $U(1)$ and $SU(3)$ can be different.
It turns out the $U(1)$ factor is an approximate global symmetry broken by the weak interactions, while the $SU(3)$ is a true, gauged, symmetry.

And what is $SU(3)$ "like"? A lot like $SU(2)$, but bigger.

13: Gauge interactions

What role can this exact $SU(3)$ play? It can be "gauged" and allow coupling of photon-like fields. Let's remember how that works.

Think about the Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + V\psi$$

We know the probability density is $\psi^*\psi$. In fact, $\psi \to e^{i\theta}\psi$ changes nothing. And $\psi \to e^{i\theta(x,t)}\psi$ doesn't change probability distribution.

But it does change the physics, because

$$\partial_t e^{i\theta(x,t)}\psi \neq e^{i\theta(x,t)}\partial_t\psi$$

Derivative can act on the phase. *Buuuuuttttt*....

14: Gauge theory: Phase freedom

What if I "fix up" the derivatives:

$$\partial_\mu \to \partial_\mu + ieA_\mu$$

with A_μ some new fields. When I change the phase, I change the A at the same time:

$$\psi \to e^{ie\theta(x,t)}\psi \qquad \text{and} \qquad A_\mu \to A_\mu - \partial_\mu\theta(x,t)$$

$$\partial_\mu\psi \to \partial_\mu e^{ie\theta(x,t)}\psi = e^{ie\theta(x,t)}(\partial_\mu\psi + ie(\partial_\mu\theta)\psi)$$

$$\text{but } (\partial_\mu + ieA_\mu)\psi \to e^{ie\theta(x,t)}(\partial_\mu\psi + ieA_\mu\psi + ie(\partial_\mu\theta)\psi - ie(\partial_\mu\theta)\psi)$$

$$= e^{ie\theta(x,t)}(\partial_\mu + ieA_\mu)\psi$$

With addition of A_μ, *covariant* derivative is now *unchanged* when I perform global phase change.
Gauge field "patches up" derivative to be *unchanged* when I make space-dependent phase transformation (gauge transformation)

15: QED again

So what is the theory of QED? Theory where I can simultaneously change overall phase on *every* field:

$$\psi_a \rightarrow e^{i\theta(x,t)eQ_a}\psi_a\,, \qquad Q = \text{charge of } \psi_a \text{ field}$$

Fix it up by $\partial_\mu \psi_a \rightarrow (\partial_\mu + ieQ_a A_\mu)\psi_a$
Gauge field changes by $A_\mu \rightarrow A_\mu - \partial_\mu \theta$

Any other exact continuous transformation symmetry could have the same modification. QCD is theory where this occurs with $SU(3)$.

16: Color Gauge Fields: Gluons

Consider (gauge) transformation, rotating between colors at every x^μ:

$$u(x^\mu) \to \exp\left(-ig_s \sum_{a=1}^{8} \theta_a(x^\mu) \frac{\lambda_a}{2}\right) u(x^\mu)$$

(and simultaneously the same transformation to d, s, c, b)

Every normal derivative is replaced with covariant derivative:

$$\gamma^\mu \partial_\mu u \to \gamma^\mu \left(\partial_\mu - ig_s \sum_a G^a_\mu \frac{\lambda_a}{2}\right) u$$

and same for d, s, c, b. And the G_μ (gluon fields) transform as

$$G^a_\mu \to G^a_\mu + \partial_\mu \theta_a + ...$$

17: How gauge fields transform

But wait!
Consider G^1_μ. It couples green to red. This is "green-anti-red" gluon field.

Perform a *spacetime-independent* rotation which trades green for blue.
Surely I will now have a G^4_μ field, "blue-anti-red"? Yes, even though θ was spacetime independent. Accounting for this requires an extra term:

$$G^a_\mu \to G^a_\mu + \partial_\mu \theta_a + f_{abc} G^b_\mu \theta_c$$

where f_{abc} are coefficients describing failure to commute:

$$\left[\lambda_a\,,\,\lambda_b\right] = 2if_{abc}\lambda_c$$

Nontrivial gauge behavior will also cause gluons to interact with each other.

18: Quantum Chromodynamics

This leads to the theory Quantum Chromodynamics. Like QED but

- Particles (quarks) have color index
- Vertices have Gell-Mann matrices λ^a.
- Eight "photon" fields G^a_μ the gluons
- Gluons interact *with each other*

19: Feynman Rules!

20: Summary

- Quarks are new charged particles which come in identical triplets
- We distinguish elements of triplet with a “color” index
- There are 5 “flavors” (also at high energies, the top quark).
- In e^+e^- annihilation, we can count the charges and colors
- Colors transform under $SU(3)$ color transformations
- $SU(3)$ is to Gell-Mann as $SU(2)$ is to Pauli matrices
- Gluons change color under gauge transform, and interact with each other
- More complicated Feynman rules.