Teilchenphysik:

Lecture 13: QCD matrix elements

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So now we know the theory QCD!

- Quarks which come in triples
- $SU(3)$ matrices for gauge freedom
- Lie algebra – Gell-Mann matrices – for couplings to...
- Eight “gluons” which play the role of photons
- New coupling g_s, with $\alpha_s = \frac{g_s^2}{4\pi}$ much larger than $\alpha = \frac{g_e^2}{4\pi} = \frac{1}{137}$
- Gluons interact *directly* with each other

What do real calculations look like in this theory?

Well, let’s try it out!

2: Questions we can ask

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- Scattering of Quark from Quark $qq \to qq$
- Quark from antiquark $q\bar{q}' \to q\bar{q}'$
- Quark from antiquark of same type
- Quark-antiquark annihilation $q\bar{q} \to gg$
- Quark-gluon scattering $qg \to qg$ and $\bar{q}g \to \bar{q}g$
- All gluons $gg \to gg$
- Higher-order processes: $qq \to qqg$ etc.

(note: each quark type has a conserved number! up-number, down-number...)

But do these even make sense, given that quarks are confined into gluey balls called hadrons?
Long story. Short version: yes within some specific context.

3: Quark on quark

Consider scattering a quark against a quark:

Pick starting colors: r and g
Each $\rightarrow$ starting column vector
Pick final colors: b and g (?)
Each $\rightarrow$ final row vector
What gluon "color" to use?
We must sum over all of them!

Result: 0 for $rg \rightarrow bg$. But for $rg \rightarrow gr$, we get $1 \times 1 + (-i) \times i = 2$...

4: Unknown starting colors

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Repeat, *averaging* initial color and *summing* final, and calculating $\mathcal{M}^*\mathcal{M}$ rather than $\mathcal{M}$:

$$\mathcal{M} = ... \sum_\alpha c_3^\dagger \frac{\lambda^\alpha}{2} c_1 \ c_4^\dagger \frac{\lambda^\alpha}{2} c_2$$

$$|\overline{\mathcal{M}}|^2 = \frac{...}{3\times 3} \sum_{c_1 c_2 c_3 c_4} \sum_{\alpha\beta} c_3^\dagger \frac{\lambda^\alpha}{2} c_1 \ c_4^\dagger \frac{\lambda^\alpha}{2} c_2 \ c_1^\dagger \frac{\lambda^\beta}{2} c_3 \ c_2^\dagger \frac{\lambda^\beta}{2} c_4$$

$$\text{but} \sum_{c_1} c_1 c_1^\dagger = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{1}$$

$$|\overline{\mathcal{M}}|^2 = \frac{...}{9} \sum_{\alpha\beta} \text{Tr} \ \frac{\lambda^\alpha}{2}\mathbf{1}\frac{\lambda^\beta}{2}\mathbf{1} \ \times \ \text{Tr} \ \frac{\lambda^\alpha}{2}\mathbf{1}\frac{\lambda^\beta}{2}\mathbf{1}\text{but}$$

$$|\overline{\mathcal{M}}|^2 = \frac{...}{36} \sum_{\alpha\beta} \delta_{\alpha\beta}\delta_{\alpha\beta} = \frac{...}{36} \times 8$$

The rest is just like $e^-\mu^- \to e^-\mu^-$

5: Quark on Antiquark

If we average over colors, scattering q from $\bar{q}$ is:

Exactly the same as for qq scattering.

6: Attractive or repulsive?

in QED the $|\mathrm{M}|^2$ for $e^-\mu^- \to e^-\mu^-$ and $e^-\mu^+ \to e^-\mu^+$ are the same at leading order.

But the $\mathcal{M}$ are opposite! Like-sign or opposite sign.
Like-sign is repulsive, opposite-sign attractive. **It matters! Bound states...**

7: Attractive or repulsive? $q\bar{q}$

The initial color state can be correlated! Consider $q\bar{q}$, but

$$c_1 c_2^\dagger = \frac{r\bar{r} + g\bar{g} + b\bar{b}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \lambda^9$$

Consider two possible *final* states: $c_4 c_3^\dagger = \lambda^9/\sqrt{2}$ or $c_4 c_3^\dagger = \lambda^3/\sqrt{2}$

$$\begin{aligned} \mathcal{M} &= \sum_\alpha c_3^\dagger \frac{\lambda^\alpha}{2} c_1 c_2^\dagger \frac{\lambda^\alpha}{2} c_4 \\ &= \sum_\alpha \mathrm{Tr}\, \frac{\lambda^\alpha}{2} \frac{\lambda^9}{\sqrt{2}} \frac{\lambda^\alpha}{2} \frac{\lambda^{9\mathrm{or}3}}{\sqrt{2}} \end{aligned}$$

Well $\lambda^9 = \sqrt{\frac{2}{3}}\mathbf{1}$. And $\sum_\alpha \lambda^\alpha \lambda^\alpha = \frac{16}{3}\mathbf{1}$.
If final state is singlet, nonzero answer – in fact, $4/3$. Same sign as $e^- \mu^+ \to e^- \mu^+$
Amplitude for "octet" final state is zero. Color conservation.

8: The other color combination

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What if the q and $\bar{q}$ have "different" colors?

$$c_1 c_2^\dagger = \text{a matrix} = \frac{1}{\sqrt{2}} c_9 \lambda^9 + \sum_\alpha c_\alpha \frac{1}{\sqrt{2}} \lambda^\alpha$$

are called *singlet* and *octet* combinations.
For the octet, c_α goes to c_β with amplitude

$$\begin{aligned} \mathcal{M}_{\alpha\beta} &= \ldots \sum_\gamma c_3^\dagger \frac{\lambda^\gamma}{2} c_1 \; c_2^\dagger \frac{\lambda^\gamma}{2} c_4 \\ &= \ldots \sum_\gamma \text{Tr} \; \frac{\lambda^\alpha}{\sqrt{2}} \frac{\lambda^\gamma}{2} \frac{\lambda^\beta}{\sqrt{2}} \frac{\lambda^\gamma}{2} \\ &= \ldots \times \; \frac{-1}{6} \delta_{\alpha\beta} \end{aligned}$$

Color combination stays same, and interactive is repulsive and 8 times weaker than for singlet channel.

9: Overall counting??

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We find that the color-averaged squared matrix element goes as:

$$|\overline{\mathcal{M}}|^2 \propto \frac{2}{9}$$

For the fundamental – 1/9 of cases – $\mathcal{M} \propto 4/3$, and for adjoint – 8/9 of cases – $\mathcal{M} \propto -1/6$. Does that work out?

$$\frac{1}{9} \times \left(\frac{4}{3}\right)^2 + \frac{8}{9} \times \left(\frac{-1}{6}\right)^2 = \frac{1}{9}\frac{16}{9} + \frac{8}{9}\frac{1}{36} = \frac{2}{9}$$

Yes the counting comes out right.

Also note: one attractive, one repulsive, “average” is neither.

10: Two quarks?

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Consider quark-on-quark scattering:

$$\mathcal{M} = \ldots \sum_{\alpha} c_3^\dagger \frac{\lambda^\alpha}{2} c_1 c_4^\dagger \frac{\lambda^\alpha}{2} c_2$$

What are possible $c_1 c_2$ combinations?
Not a matrix. Two column vectors. Two possibilities:

- Symmetrize: $c_1 c_2 = c_2 c_1$. 6 independent choices
- Antisymmetrize: $c_1 c_2 = -c_2 c_1$. 3 independent choices.

Call the matrix elements $\mathcal{M}_6$ and $\mathcal{M}_{\bar{3}}$.

Cheaters' way: on average no net attraction or repulsion:

$$3\mathcal{M}_{\bar{3}} + 6\mathcal{M}_6 = 0 \qquad \text{and} \qquad \frac{3}{9}\mathcal{M}_{\bar{3}}^2 + \frac{6}{9}\mathcal{M}_6^2 = \frac{2}{9}$$

So $\mathcal{M}_6 = 1/3$ (repulsive) and $\mathcal{M}_{\bar{3}} = -2/3$ (attractive)

11: When does this matter?

Suppose two different hadrons scatter.
A q from one hadron hits a q or $\bar{q}$ in the other.
There is no reason their colors should be related. Color average.

But if a q and $\bar{q}$ are *bound together* in a hadron,
then their colors certainly *can* be correlated.

We see that the color-singlet combination is attractive.
So $q\bar{q}$ systems should be color-singlet bound-state systems: **Mesons**

And two quarks? The antisymmetric combination "acts like" an antiquark in terms of color: $rg - gr \to \bar{b}$, $br - rb \to \bar{g}$, $gb - bg \to \bar{r}$. Attractive to a third quark, leading to colorless 3-quark system: **Baryons**

$$u_{1a} u_{2b} u_{3c} \propto \epsilon_{abc}$$

12: Quark-gluon scattering?

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Now there are three diagrams!

Group theory is doable but *no longer simple*

13: Gluon external states, polarization

Think about the squared matrix element for these diagrams:

In computing $|\mathcal{M}_1 + \mathcal{M}_2|^2$, we can replace $\sum_\lambda \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) \to -g_{\mu\nu}$ as discussed. But with $\mathcal{M}_3$, this *does not work.* Need to do polarization sum.

Computing the quarks-only processes is straightforward.
The processes with gluons are *not* straightforward. But doable...

$$|\overline{\mathcal{M}}|^2_{q_1 q_2 \to q_1 q_2} = \frac{4g_s^4}{9}\frac{s^2+u^2}{t^2}$$

$$|\overline{\mathcal{M}}|^2_{q_1 \bar{q}_1 \to q_2 \bar{q}_2} = \frac{4g_s^4}{9}\frac{t^2+u^2}{s^2}$$

$$|\overline{\mathcal{M}}|^2_{q_1 q_1 \to q_1 q_1} = \frac{4g_s^4}{9}\left(\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}\right) - \frac{8g_s^4}{27}\frac{s^2}{tu}$$

$$|\overline{\mathcal{M}}|^2_{q_1 \bar{q}_1 \to gg} = \frac{32}{27}\frac{u^2+t^2}{ut} - \frac{8g_s^4}{3}\frac{u^2+t^2}{s^2}$$

$$|\overline{\mathcal{M}}|^2_{qg \to qg} = -\frac{4g_s^4}{9}\frac{u^2+s^2}{us} + \frac{u^2+s^2}{t^2}$$

$$|\overline{\mathcal{M}}|^2_{gg \to gg} = \frac{9g_s^4}{2}\left(3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{st}{u^2}\right)$$

For $gg \to q\bar{q}$ you have to switch which colors you average and which you sum, so it will have $1/64$ instead of $1/9$. Otherwise same.

Nowadays people calculate 1-loop and 2-loop squared matrix elements with extra outgoing gluons, using computer algebra and super-clever symmetry arguments

15: Loop effects?

Consider scattering beyond the leading order:

As in QED, coupling varies with scale. Unlike QED, it shrinks in UV!

$$\frac{d\alpha}{d\ln(p^2)} \equiv \beta_\alpha = \frac{1}{3\pi}\alpha^2 \quad \text{but} \quad \frac{d\alpha_s}{d\ln(p^2)} = \frac{1}{12\pi}(2N_f - 33)\alpha_s^2$$

where N_f is the number of quark flavors and -33 comes from gluon self-interaction.

16: Asymptotic Freedom

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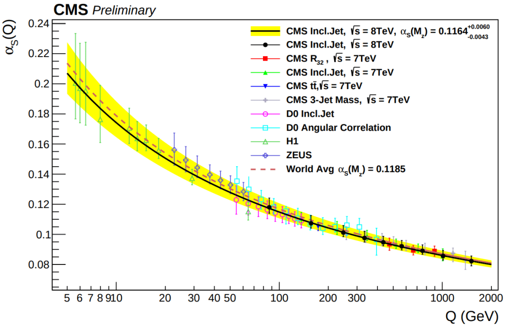
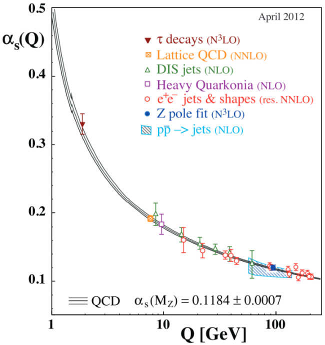
As energy scale p^2 increases, α_s gets smaller!





Good agreement with experiment. But a problem at low energies.

17: What happens at large distances?

At low energies / large distances coupling is *large*.
Occurs around 300 MeV or 0.7 fermi.
Expect bound states even relativistically.

This will be the topic for next time.

18: Summary

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- Scattering is very color dependent.
- Use explicit matrices to calculate specific in, out colors.
- Uncorrelated quarks (from 2 hadrons..) – do color averaging
- Correlated quarks (living in one hadron) – think about overall color state.
- $q\bar{q}$: singlet or octet. Strongly attractive, weakly repulsive
- qq: triplet or sextet. Attractive or repulsive.
- Quark-gluon scattering: calculations get messy, polarization trick no longer works
- Loop effects: asymptotic freedom
- Long distance / low momentum: coupling is large. Bound states...