Teilchenphysik:

Lecture 14: Bound States

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This lecture will be two somewhat disjoint parts.

First: why do we expect bound states in QCD,
and how should we think about them?

Second: how do bound states work in an example we understand, QED?

A more detailed discussion of QCD bound states will have to wait for January.

2: The idea of potential energy

Consider an electron e^- and a proton p^+. Two approaches.
Approach 1: perturb in the EM interaction. Diagrams:

At low velocities I have to resum the ladders. I get Schrödinger Eq.
(No, it's not easy to show, and we will not try.)

3: Potential energy II

The other approach is that I *don't* perturb in A^0.
I treat p as at rest (CM coord, reduced mass), solve for A^0, solve e^- fully in A^0 BG
But I perturb in A_i the vector potential, suppressed by v/c.

Unperturbed solution for A^0: $A^0 = -\alpha/r$. Electron solves Dirac equation with A^0:

$$
\begin{aligned}
(i\gamma^\mu D_\mu - m)\psi &= 0 \\
i\gamma^\mu(\partial_\mu + ieA_\mu)\psi &= m\psi \\
i\gamma^\mu \partial_\mu \psi &= m\psi + A_0\gamma^0\psi = \left(m - \frac{\alpha}{r}\gamma^0\right)\psi
\end{aligned}
$$

This is relativistic generalization of Schrödinger:
the Dirac equation for an electron in a hydrogen atom.

4: Back to QCD

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Consider a quark and an antiquark. To make the argument simpler, assume for now that *at least one* is heavy.
Similar situation as QED: treat G_0 explicitly, it is a potential.
We already saw that for singlet color combination, it's attractive:

$$V(r) = -\frac{4\alpha_s}{3r}, \qquad \alpha_s = \frac{g_s^2}{4\pi} \geq \frac{1}{8}$$

But what value of α_s should be used? It's energy dependent!

Very good question. Answer, roughly:

- For $r < \frac{\hbar c}{500\text{ MeV}} \sim 0.4$ fm, use α_s at energy scale $E = \hbar c/r$.
- For longer distances – not obvious what you should do.
- Indications (models, lattice QCD, etc): $V(r)$ rises nearly linearly at larger distances! (QCD string ...)

5: Cartoon picture of effective potential

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Roughly speaking, potential is $-4\alpha/3r$ at short distance, σr at large:

Confinement: q and $\bar{q}$ can *never* escape.

6: String breaking

Existence of light quarks changes that.
It doesn't cost much energy to generate a new $q\bar{q}$ pair:

Pull q and $\bar{q}$ apart: new ones appear in between, and you get two (or more) isolated hadrons.

7: QCD: What we expect

- Potential is $V \sim -1/r$ at short range, $V \sim r$ at long range
- For heavy quarks forming tight bound states, should be a lot like hydrogen
- For light quarks exploring nearly linear part, should be a little like hydrogen (spherical harmonics and radial functions) but very different energy levels

Exact calculations are hard.
Qualitative picture less hard. Sort of like hydrogen.

So let's start by reviewing what happens with hydrogen!

8: Hydrogen: Dirac equation

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Dirac eq. for hydrogen: $$i\gamma^0\partial_0\psi = \left(-i\gamma^i\nabla^i + \frac{mc}{\hbar} - \frac{\alpha\gamma^0}{r}\right)\psi$$

There are two approaches.

- Dirac's way: directly solve it. Doable, but somewhat harder than
- Nonrelativistic approximation: assume $mc \gg \alpha\hbar/r$.
  Perturb in $mc \gg \hbar\alpha/r$ and upper ψ components $\gg$ lower components.

Of course Dirac's way is "right."

But we want to see how Schrödinger equation emerges from this,
so we will pursue the nonrelativistic approximation way.

Also since proton is static, $i\partial_0 \to E/\hbar$.
And I will stop writing $\hbar$ and c now.

9: Very lowest order

$$\gamma^0 E\psi = m\psi \qquad \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & -E \end{bmatrix} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix} = m \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}$$

Two solutions with $E = m$: first and second entry.
Two solutions with $E = -m$: I don't want those.

I should work in terms of blocks: the upper two components, and the lower two components

At lowest order, only the upper elements of ψ are nonzero

10: Next order: lower entries

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The gradient term requires that the bottom enties *not* be zero: including the $-i\gamma^i\nabla^i$ term, we find, in 2×2 blocks:

$$\begin{bmatrix} E-m-V & 0 \\ 0 & -E-m-V \end{bmatrix} \begin{bmatrix} \psi_{\text{up}}(x) \\ \psi_{\text{down}}(x) \end{bmatrix} = \begin{bmatrix} 0 & -i\sigma^i\nabla^i \\ i\sigma^i\nabla^i & 0 \end{bmatrix} \begin{bmatrix} \psi_{\text{up}}(x) \\ \psi_{\text{down}}(x) \end{bmatrix}$$

The lower component is the equation

$$(-E-m-V)\psi_{\text{down}}(x) = i\sigma^i\nabla^i\psi_{\text{up}}(x)$$

We can approximate $-E-m-V = -2m$ here. The upper equation is

$$(E-m-V)\psi_{\text{up}}(x) = -i\sigma^i\nabla^i\psi_{\text{down}}(x)$$

Inserting the approximated first equation:

$$E\psi_{\text{up}}(x) \simeq (m+V)\psi_{\text{up}}(x) - \frac{1}{2m}\sigma^i\sigma^j\nabla^i\nabla^j\psi_{\text{up}}(x)$$

11: Schrödinger equation emerges

$$E\psi_{\mathrm{up}}(x) \simeq (m + V)\psi_{\mathrm{up}}(x) - \frac{1}{2m}\sigma^i\sigma^j\nabla^i\nabla^j\psi_{\mathrm{up}}(x)$$

The derivatives commute, so

$$\sigma^i\sigma^j\nabla^i\nabla^j = \frac{\sigma^i\sigma^j + \sigma^j\sigma^i}{2}\nabla^i\nabla^j = \delta_{ij}\nabla^i\nabla^j = \nabla^2$$

and we get the Schrödinger equation!

$$E\psi(x) = \left(m + V(r) - \frac{1}{2m}\nabla^2\right)\psi$$

However

- ψ is now two-component
- Had there been $\vec{A}$ and magnetic fields, things would be different

12: Magnetic field?

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When there's a magnetic field: $\nabla^i \to D^i = \nabla^i + ig_e A^i$, A^i the vector potential.
Now derivatives don't commute: $2AB = (AB + BA) + (AB - BA)$, so

$$\begin{aligned}
\sigma^i \sigma^j D^i D^j &= \frac{1}{4}\left(\left\{\sigma^i\,,\,\sigma^j\right\} + \left[\sigma^i\,,\,\sigma^j\right]\right)\left(\left\{D^i\,,\,D^j\right\} + \left[D^i\,,\,D^j\right]\right) \\
&= \frac{1}{4}\left((2\delta_{ij} + 2i\epsilon_{ijk}\sigma_k)\right)\left((D_i D_J + D_j D_i) - ig_e(\partial_i A_j - \partial_j A_i)\right) \\
&= D^2 + g_e \epsilon_{ijk}\frac{\sigma_k}{2} F_{jk} \\
&= D^2 + g_e \sigma_k B_k = D^2 + 2g_e S_k B_k
\end{aligned}$$

This is *twice* the spin-dot-magnetic effect you might have expected.

- There is an interaction between spin and magnetic fields!
- It is twice as strong "as expected" with g-factor 2.

13: Back to Schrödinger

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Write ∇^2 in terms of L^2 and a radial part:

$$\nabla^2\psi = \left(\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}L^2\right)\psi$$

Look for eigenstates of L^2 with definite ℓ: $L^2 \rightarrow \ell(\ell+1)$ with $\ell = 0, 1, 2, 3, \ldots$ (spdf)
Solve each radial equation, finding

$$E = mc^2 - \frac{\alpha^2}{2n}mc^2\,, \qquad n = (\ell+1), (\ell+2), (\ell+3), \ldots$$

The fact that our particle has spin has not yet entered in discussion, nor are there relativistic effects at this point.

There is an accidental degeneracy between $2s = 2p$, $3s = 3p = 3d$, etc.

14: Relativistic corrections

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We took two exact equations:

$$(E - m - V)\psi_{\text{up}}(x) = -i\sigma^i\nabla^i\psi_{\text{down}}(x), \qquad (-E - m - V)\psi_{\text{down}}(x) = i\sigma^j\nabla^j\psi_{\text{up}}(x)$$

and then approximated $E + V = m$ in the latter. If we don't, we get

$$(E - m - V)\psi_{\text{up}}(x) = -i\sigma^i\nabla^i\frac{1}{-E - m - V}i\sigma^j\nabla^j\psi_{\text{up}}(x)$$

Here $E \neq m$ and the ∇^i can act on V.
This gives rise to relativistic corrections including spin-orbit coupling.

The corrections are down by another power of α^2:

$$\Delta E_{nj} = -\alpha^4 mc^2\frac{1}{4n^4}\left(\frac{2n}{j + \frac{1}{2}} - \frac{3}{2}\right) .$$

Still some degeneracy: $2s$ has $j = 1/2$, $2p$ has $j = 1/2, 3/2$...

15: Higher corrections

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We could work harder and find α^6 corrections.
BUT we would be missing something – quantum fluctuations in $\vec{A}$.
These take *real work* to compute and enter at $\alpha^5 mc^2$.

Also the proton has spin, and with it a dipolar magnetic field.
So there is also spin-spin coupling. It's proportional to

$$\Delta E_{ss} \sim \alpha^4 mc^2 \frac{m_e}{m_p}$$

suppressed by the "large" proton mass. The 1*s* Hydrogen state has a spin-$\frac{1}{2}$ e^- and spin-$\frac{1}{2}$ p^+ and can therefore be either spin-0 or spin-1.
Spin-0 is lower energy. The splitting is 21cm....

16: What about e^+e^-??

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When e^+ and e^- meet, they don't instantly annihilate.
First they stick together into an "atom" **Positronium**.

At the crudest level: Schrödinger with reduced mass $\mu = m_e/2$.

$$E = 2mc^2 - \frac{\alpha^2}{4n^2}mc^2\,, \qquad \ell = 0, 1, 2, \ldots \qquad n = (\ell+1), (\ell+2), \ldots$$

Relativistic corrections are different because e^+

- has a way bigger magnetic moment
- moves as much as the e^- does

Spin-spin effects are now part of the fine structure, $\sim \alpha^4$.

17: Positronium

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We need to talk about positronium because it's a good *analogue* for what can happen with two quarks.
To find the energy levels, do following:

- First find $\ell = 0, 1, 2, \ldots$
- Allowed n values are $n = (\ell + 1), (\ell + 2), \ldots$
- *Next* combine the spins. The total spin can be $S_{\text{tot}} = 0$ or $S_{\text{tot}} = 1$
- *Next* combine S and L to get J values.

$$E = 2mc^2 - \frac{\alpha^2 mc^2}{4n^2} + \frac{\alpha^4 mc^2}{2n^3}\left[\frac{11}{32n} - \frac{2+\epsilon}{2(2\ell+1)}\right], \quad \epsilon = \begin{cases} 0 & s = 0 \\ \frac{-(3\ell+4)}{(\ell+1)(2\ell+3)} & s = 1, j = \ell + 1 \\ \frac{1}{\ell(\ell+1)} & s = 1, j = \ell \\ \frac{3\ell-1}{\ell(2\ell-1)} & s = 1, j = \ell - 1 \end{cases}$$

Plus one more effect!

18: Virtual pair annihilation

If $\ell = 0$ the e^{+}, e^{-} meet.
If $j = 1$, they have the same quantum numbers as a photon. So

This virtual process is allowed. Shifts energy *up* by

$$\Delta E_{\text{ann}} = \alpha^4 mc^2 \frac{1}{4n^3} \delta_{\ell 0} \delta_{j1}$$

19: Positronium decays!

The e^- and e^+ need to “collide” which requires $\ell = 0$
The C of initial and final states must be same:

$$C_{\text{positronium}} = (-1)^{\ell+s} = (-1)^s\,, \qquad C_{n_\gamma\,\text{photons}} = (-1)^{n_\gamma}$$

For $s = 0$ we need n_γ even (2). For $s = 1$ we need n_γ odd (3).
Decay goes as α^3 to get the e^-, e^+ to find each other, times α^2 or α^3 from annihilation diagram:

- Width of $s = 0$ (para-positronium) is $\Gamma = \alpha^5 m_e c^2/2$
- Width of $s = 1$ (ortho-positronium) is $\Gamma \sim \alpha^6 m_e c^2$

Later it will be important that the $s = 1$ state is much longer lived.

20: Summary

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- Bound states arise because of an attractive *potential energy*
- For QCD we expect singlet $q\bar{q}$ to be attractive with $\sim -1/r$ short-distance and $\sim r$ large-distance behavior
- This leads to **confinement**: new $q\bar{q}$ pairs appear when $q, \bar{q}$ move away from each other, forming 2+ colorless objects.
- For a feel for what happens, we revisit *hydrogen* and *positronium*
- We see how *Schrödinger* emerges from *Dirac*, and how high-order corrections also arise
- For hydrogen (heavy-light), spin-orbit is important but spin-spin is a small effect.
- Fine structure in positronium depends on spin-spin, not just j, ℓ
- The tightest-bound state is $s = 0, j = 0$
- The $j = 0$ decays to 2γ, $j = 1$ to 3γ much more slowly.