Teilchenphysik:

Lecture 16: Lie groups and Lie algebras

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This lecture will consist entirely of a long “aside” or review about Lie stuff.

- What is a Lie group, concretely?
- How to think about them
- Compact vs noncompact
- Lie algebra, exponentiation to get the group
- Higher representations and their meaning

2: Lie groups as matrices

Every Lie group can be viewed as a *family of matrices*.
Group operation is matrix multiplication. Choose a set of matrices which is closed under multiplication and contains an identity and inverses.
"All $N \times N$ matrices doesn't work – not all invertible. Need some added rule.

- $GL(N, \mathcal{C})$: $N \times N$ complex invertible matrices
- $SO(N)$: $N \times N$ real determinant-1 matrices M obeying $M^\top = M^{-1}$ or equivalently $M^\top_{ij} \delta_{jk} M_{kl} = \delta_{il}$
- $SU(N)$: $N \times N$ complex determinant-1 matrices M obeying $M^\dagger = M^{-1}$
- $SO(N, M)$: if g_{ij} is diagonal with N $(+1)$ entries and M (-1) entries, $(N + M) \times (N + M)$ matrices with $M^\top_{ij} g_{jk} M_{kl} = g_{il}$
- $SP(N)$, G_2, F_4, E_6, E_7, E_8: something-or-other

3: Continuous: neighborhood of the identity

Mathematicians can define a metric on space of matrices.
Group has identity element. Pick neighborhood with everything less than ϵ away from identity.

This is a “tiny ball” of group elements. Homeomorphic to a *d*-dimensional ball, where *d* is the **dimension of the group**

The whole group will be a *d*-dimensional **manifold**.

4: An example: $SU(2)$

$SU(2)$ are 2×2 complex matrices U with $U^\dagger = U^{-1}$.

$$U = c_0 \mathbf{1} + i c_1 \tau_1 + i c_2 \tau_2 + i c_3 \tau_3\,, \qquad \tau_i \text{ Pauli matrices}, \qquad c_0^2 + c_1^2 + c_2^2 + c_3^2 = 1$$

$$\begin{aligned}
U^\dagger U &= (c_0 \mathbf{1} - i c_i \tau_i)(c_0 \mathbf{1} + i c_j \tau_j) \\
&= c_0^2 \mathbf{1} + \sum_{ij} c_i c_j \tau_i \tau_j \\
&= c_0^2 \mathbf{1} + \frac{1}{2} \sum_{ij} c_i c_j (\tau_i \tau_j + \tau_j \tau_i) \\
&= (c_0^2 + c_1^2 + c_2^2 + c_3^2)\mathbf{1} = \mathbf{1}
\end{aligned}$$

Write it out long-hand:

$$\text{Most general } SU(2) \text{ is:} \quad U = \begin{bmatrix} c_0 + i c_3 & c_2 + i c_1 \\ -c_2 + i c_1 & c_0 - i c_3 \end{bmatrix}, \quad c_0^2 + c_i^2 = 1$$

5: $SU(2)$ interpretation

Think of 4-dimensional space of (c_0, c_1, c_2, c_3) values. $SU(2)$ is 3-sphere

6: $SU(2)$: neighborhood of north pole

7: $SU(2)$: how to get to a general point

Consider some point
Find arc length from N-pole
Find direction
Name arc-length $|\theta|$
Name direction $\hat{\theta}$
$\vec{\theta} = \theta_i$ is 3-vector
$(c_1, c_2, c_3) = \hat{\theta} \sin|\theta|$
$c_0 = \cos|\theta|$
To get to (c_0, c_1, c_2, c_3),
pick $|\theta| = \mathrm{Arccos}(c_0)$
and pick $\hat{\theta} = (c_1, c_2, c_3)/\sqrt{c_i^2}$

8: Group $SU(2)$ is compact!

Pick a direction
Vary length $|\theta|$
Get farther away
until $|\theta| = \pi$
Reach south pole
$|\theta| > \pi$: pass pole
$|\theta| = 2\pi$: back to N-pole!

9: Exponentiation

How exactly do I turn $\vec{\theta}$ into $SU(2)$ element?
Consider $|\theta|$ tiny: $c_0 \simeq 1$ and

$$U = \cos|\theta|\mathbf{1} + i\sin|\theta|\hat{\theta}_i\tau_i \simeq \mathbf{1} + i\theta_i\tau_i$$

The $i\tau_i$ are directions I can travel and θ_i are distances to go in those directions.
Finite distance: **exponentiation**

$$U = \exp(i\theta_i\tau_i) \equiv \mathbf{1}\sum_{n=0}^{\infty}\frac{\left(i\sum_i\theta_i\tau_i\right)^n}{n!} = \mathbf{1} + i\theta_i\tau_i - \frac{1}{2}\theta_i\theta_j\tau_i\tau_j - \frac{i}{6}\theta_i\theta_j\theta_k\tau_i\tau_j\tau_k + ...$$

Use repeatedly $\sum_{ij}\theta_i\theta_j\tau_i\tau_j = |\theta|^2$:

$$\begin{aligned} U &= \mathbf{1}\left(1 - \frac{|\theta|^2}{2!} + \frac{|\theta|^4}{4!}...\right) + i\theta_i\tau_i\left(1 - \frac{|\theta|^2}{3!} + \frac{|\theta|^4}{5!}...\right) \\ &= \cos|\theta|\mathbf{1} + \sin|\theta|\hat{\theta}_i i\tau_i \end{aligned}$$

10: What's general to all Lie groups?

A Lie group *always* has

- a d-dimensional neighborhood around the identity.
  d is the dimension of the group.
- a basis of d orthonormal directions T^a – the Lie algebra
  (for $SU(2)$, $\tau_i/2$)
- Any element[1] can be found by exponentiating (i times) Lie algebra elements times finite lengths

Cataloguing groups requires cataloguing Lie algebras. Need to know:

1. the dimension d and Lie elements T^a
2. the commutation relations $[T^a, T^b] = if_{abc}T^c$

determines the whole group.

[1] up to discrete structure

11: even simpler example: $U(1)$

consider 1×1 complex matrices – just complex numbers!
Group $U(1)$ is unit-length complex numbers

Most general form: $e^{i\theta}$. $\theta \in \mathcal{R}$ but θ and $\theta + 2\pi n$ equivalent.

Close to identity: $1 + i\epsilon$. Lie algebra: 1 direction, generated by the number i.

Exponentiation: most general element is $\exp(i\theta)$ with i the "direction" and θ the "distance"

12: Not all groups are compact!

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Consider $SO(1,3)$ Lorentz transforms. 4×4 matrices...
Small "boost" in *x*-direction has:

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here β is distance, that matrix is Lie-algebra direction (there are a total of 6 possible Lie-algebra direction matrices)
Exponentiation yields:

$$\exp \beta \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cosh(\beta) & -\sinh(\beta) & 0 & 0 \\ -\sinh(\beta) & \cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This gets ever further from the identity.

This group is *noncompact*.

Internal symmetries *must be compact.*

13: The group $SU(3)$

The group $SU(3)$ is 3×3 complex matrices with $U^\dagger = U^{-1}$ and with unit determinant.
Consider an element *very close to the identity*:

$$U = \mathbf{1} + i\epsilon + \mathcal{O}(\epsilon^2)$$

Here ϵ is a *matrix* of small numbers. We need

$$U^\dagger U = (\mathbf{1} - i\epsilon^\dagger)(\mathbf{1} + i\epsilon) = \mathbf{1} + i(\epsilon - \epsilon^\dagger) + \mathcal{O}(\epsilon^2)$$

to be the identity: so $\epsilon - \epsilon^\dagger = 0$ or $\epsilon = \epsilon^\dagger$.

Put another way, ϵ must be Hermitian. Also

$$\mathrm{Det} U \simeq 1 + i \,\mathrm{Tr}\, \epsilon = 1 \qquad \Rightarrow \qquad \mathrm{Tr}\, \epsilon = 0$$

14: The group $SU(3)$ part 2

Now consider ϵ a *finite* Hermitian matrix. Consider

$$U = \exp[i\epsilon] \equiv \sum_{n=0}^{\infty} \frac{(i\epsilon)^n}{n!} \qquad \text{and its dagger} \qquad U^\dagger = \sum_{n=0}^{\infty} \frac{(-i\epsilon)^n}{n!} = \exp[-i\epsilon]$$

The product is:

$$\begin{aligned} U^\dagger U &= \sum_{n=0}^{\infty} (i\epsilon)^n \times \sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \\ &= \sum_{n=0}^{\infty} \frac{(i\epsilon)^n}{n!} \times \sum_{m=0}^{n} (-1)^m \frac{n!}{m!(n-m)!} = \sum_{n=0}^{\infty} \frac{(i\epsilon)^n}{n!} \times (1-1)^n = \mathbf{1} \end{aligned}$$

so the exponential of i times a finite Hermitian matrix is in $SU(3)$.

15: Lie algebra generators

What's the most general Hermitian matrix ϵ?
Useful to build a *orthonormal basis* of traceless Hermitian matrices
Most general value is a linear combination with arbitrary coefficients.

For $SU(2)$ there were $2^2 - 1 = 3$ independent matrices: Pauli matrices
For $SU(3)$ there are $3^2 - 1 = 8$ independent matrices: Gell-Mann matrices

So, *is* $\exp(ic^\alpha \lambda^\alpha) = \cos|c|\mathbf{1} + \sin|c|\hat{c}^\alpha \lambda^\alpha$ like for $SU(2)$?
Sadly, no. Because $\lambda^\alpha \lambda^\beta + \lambda^\beta \lambda^\alpha \neq 2\delta^{\alpha\beta}\mathbf{1}$.

I *can* get the most general $SU(3)$ element by exponentiation.
But there is no beautiful interpretation as a sphere in 9 dimensions, as there was for $SU(2)$.

16: Do orbits always close in $SU(3)$?

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No they don't. Consider

$$\exp\left(ia\lambda^3 + ib\sqrt{3}\lambda^8\right) = \exp\begin{bmatrix} i(b+a) & 0 & 0 \\ 0 & i(b-a) & 0 \\ 0 & 0 & -2ib \end{bmatrix} = \begin{bmatrix} e^{i(a+b)} & 0 & 0 \\ 0 & e^{i(b-a)} & 0 \\ 0 & 0 & e^{-2ib} \end{bmatrix}$$

This only returns to the identity if $m(b+a) = n(b-a) = 2\ell b$ for (ℓ, m, n) three integers. Generically $(a+b), (b-a), 2b$ are irrationally related and the orbit never closes.

Note that λ^3 and λ^8 commute.
The dimension of maximal commuting subset of Lie algebra is the **rank**.
The rank of $SU(N)$ is $N-1$. For $SO(N)$ it's floor$(N/2)$.

17: Interpretation: torus

The subgroup you reach by exponentiating $a\lambda^3 + b\lambda^8$ is geometrically a torus.

Orbits don't close, but they don't get farther and farther away. Group is compact.

18: Representations

A **Representation** is an embedding of group G into matrices.
Requirement: if U_1, U_2 represented by M_1, M_2, then $(U_1 U_2)$ represented by $(M_1 M_2)$.
A field is *acted upon* by a group representation: $\psi \to M\psi$...

Simplest example: the fundamental representation: matrices = group itself

$$U \in G, \quad \text{transformation under } U \text{ is} \quad \psi \to U\psi$$

Antifundamental representation:

$$U \in G, \quad \text{transformation under } U \text{ is} \quad \psi \to U^*\psi$$

if ψ is in fundamental, its antiparticle $\overline{\psi}$ is in antifundamental.

19: Antifundamental and $SU(2)$

For $SU(2)$, fundamental and antifundamental are related!

$$q = \begin{bmatrix} u \\ d \end{bmatrix} \to Uq \quad \text{then} \quad \overline{q} = \begin{bmatrix} \overline{u} \\ \overline{d} \end{bmatrix} \to U^* \overline{q}$$

But introducing $\epsilon = i\sigma_2$ we find that

$$\tilde{q} = \begin{bmatrix} -\overline{d} \\ \overline{u} \end{bmatrix} = \epsilon \overline{q} \to \epsilon U^* \overline{q} = U \epsilon \overline{q} = U \tilde{q}$$

Swapping $\overline{u}, \overline{d}$ and adding a minus sign gives the same transformation properties as the quarks.

20: Antifundamental of $SU(3)$

The same is *not* true of $SU(3)$. Antifundamental is really different.

- $q\bar{q}$ can be colorless. Indeed, $q\bar{q}$ can combine into 8-element *adjoint* or 1-element *singlet* (colorless)
- qq cannot be colorless. They combine into 6 (symmetric) or $\bar{3}$ (antisymmetric).
- Add one more quark: qqq can combine into 10, 8, 8, or 1

How do I figure this out?
We need to discuss *products of representations*

But I don't think we have time to do so today.

21: Summary

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- **Lie groups** can be thought of as continuous families of matrices.
- Our favorite, $SU(2)$, has the topology of the 3-sphere
- Useful to consider the neighborhood near the identity.
- There are d directions (d=**dimension** of group) in which to leave the identity element
- Directions define the **Lie algebra** of group
- Generic group elements reached by starting at identity and propagating a *finite distance* in some initial direction
- **matrix exponentiation** performs exactly this.
- Not all Lie groups are **compact**, but those describing internal symmetries must be.
- The group $SU(3)$ is a lot like $SU(2)$, but some special $SU(2)$ properties don't work in $SU(3)$
- In particular, antifundamental representation of $SU(2)$ is equivalent to fundamental. For $SU(3)$ it is not.