Teilchenphysik:

Lecture 2: Relativity Review

TECHNISCHE UNIVERSITÄT DARMSTADT

Here we go through the relativity basics which we will need repeatedly during the rest of the course.

- Reminder: rotations and Galilean transformations
- Lorentz transformations
- 4-vector notation
- Repeated transformations, products, addition of velocities
- Invariants and the metric
- Tensors and their transformation properties

2: Reminder: rotations

Rotation by angle θ about the z axis:

$$x' = \cos(\theta)x - \sin(\theta)y$$
$$y' = \sin(\theta)x + \cos(\theta)y$$
$$z' = z$$

It is very convenient to write x, y, z and x', y', z' as vectors:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3: Repeated rotations

Product of two rotations about two axes:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta)\cos(\xi) & \sin(\theta)\sin(\xi) \\ \sin(\theta) & \cos(\theta)\cos(\xi) & -\cos(\theta)\sin(\xi) \\ 0 & \sin(\xi) & \cos(\xi) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x_i' = R_{ij} R'_{jk} x_k = R''_{ik} x_k$$

some rotation about *some* axis.
Quick way to make sure that this is really a rotation matrix?

How was I so sure that R_{ij} and R'_{jk} were rotations?

4: Invariant length

A rotation is something which *automatically preserves vector length*

$$|x|^2 \equiv x_i \delta_{ij} x_j \quad \text{must equal} \quad |x'|^2 \equiv x'_i \delta_{ij} x'_j$$

for any choice of the vector x_i. Therefore:

$$x'_i = R_{ij} x_j \;\Rightarrow\; R_{ik} x_k \delta_{ij} R_{jm} x_m = x_i \delta_{ij} x_j \quad \Rightarrow \quad x_k R^\top_{ki} \delta_{ij} R_{jm} x_m = x_i \delta_{ij} x_j$$

Here $R^\top$ is the transpose, $R^\top_{ji} = R_{ij}$. Let's rename $i \leftrightarrow k$ and $j \leftrightarrow m$

$$x_i \left(R^\top_{ik} \delta_{km} R_{mj} \right) x_j = x_i \left(\delta_{ij} \right) x_j \quad \text{for all } x_i \quad \Rightarrow \quad R^\top_{ik} \delta_{km} R_{mj} = \delta_{ij}$$

The product of $R^\top$ and R, as matrices, is the identity.

The rotation matrices are precisely[1] the 3×3 matrices which have this property; the so-called orthogonal matrices $R \in SO(3)$

[1] We also need Det $R = 1$ so that we don't mirror-image space (parity)

5: Galilean Invariance

If r' is moving at velocity $\vec{v} = (v, 0, 0)$ with respect to r then the r' coordinates have a time-dependent shift compared to the r coordinates:

$$t' = t, \quad x' = x - vt, \quad y' = y \quad z' = z \quad \text{or} \quad \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

(Why the $-$ sign?) This is a Galilean transformation, a symmetry (canonical transformation) of classical mechanics.

6: Relativity: Lorentz Transform

As you know, there are v^2/c^2 corrections to the Galilean transform. Time and x actually mix with each other!

Defining $\gamma = (1 - v^2/c^2)^{-1/2}$, one finds:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v/c^2 & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

7: Relativity and Units

Better to use ct instead of t and $\beta = v/c$, so matrix is dimensionless:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

- Time dilation
- Relativity of Simultaneity
- Motion
- Length contraction

8: 4-vector, matrix notation

TECHNISCHE UNIVERSITÄT DARMSTADT

For rotations it was super useful to use vectors x_i.
Including time, we call it a 4-vector x^μ: $x^0 = ct$, $x^1 = x_1 = x$, $x^2 = x_2 = y$, $x^3 = x_3 = z$
Upper indices mean column indices; lower indices are row indices

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \text{or} \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

The Lorentz transformation matrix has a lower and an upper index. Products:

$$x''^\mu = \Lambda'^\mu{}_\nu x'^\nu = \Lambda'^\mu{}_\nu \Lambda^\nu{}_\alpha x^\alpha$$

You can contract an upper with a lower index (row with column).
You cannot contract two upper or two lower indices (yet).

9: Greek letters...

The time component is 0 not 4. (Why?)
Roman indices *ijklmn* are space indices, $i = 1, 2, 3$
Greek indices $\mu\nu\alpha\beta$ are spacetime indices $\mu = 0, 1, 2, 3$
Use Greek letters in the order $\mu, \nu, \alpha, \beta, \gamma, \delta, \sigma, \rho$...

αA alpha	βB beta	$\gamma\Gamma$ gamma	$\delta\Delta$ delta
ϵE epsilon	ζZ zeta	ηH eta	$\theta\Theta$ theta
ιI iota	κK kappa	$\lambda\Lambda$ lambda	μM mu
νN nu	$\xi\Xi$ xi	oO omicron	$\pi\Pi$ pi
ρR rho	$\sigma\Sigma$ sigma	τT tau	$\upsilon\Upsilon$ upsilon
$\phi\Phi$ phi	χX chi	$\psi\Psi$ psi	$\omega\Omega$ omega

Avoid using iota, omicron, pi, upsilon.

10: Two Lorentz's in the same direction

TECHNISCHE
UNIVERSITÄT
DARMSTADT

Product of two Lorentz transformations, both in x-direction:

$$\begin{bmatrix} \gamma' & -\gamma'\beta' & 0 & 0 \\ -\gamma'\beta' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma'\gamma(1+\beta'\beta) & -\gamma'\gamma(\beta'+\beta) & 0 & 0 \\ -\gamma'\gamma(\beta'+\beta) & \gamma'\gamma(1+\beta'\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Same as a Lorentz transformation with velocity

$$\beta_{\text{tot}} = \frac{\beta+\beta'}{1+\beta\beta'} \quad \text{or} \quad v_{\text{tot}} = \frac{v'+v}{1+vv'/c^2}$$

What adds trivially is the rapidity $y = \operatorname{atanh}(\beta)$: $y_{\text{tot}} = y' + y$.

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cosh(y) & -\sinh(y) & 0 & 0 \\ -\sinh(y) & \cosh(y) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11: Which matrices are valid Lorentz matrices?

TECHNISCHE UNIVERSITÄT DARMSTADT

An example: the product of two valid Lorentz matrices

$$\begin{bmatrix} \gamma' & 0 & -\gamma'\beta' & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma'\beta' & 0 & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \gamma'\gamma & -\gamma'\gamma\beta & -\gamma'\beta' & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\gamma'\gamma\beta' & \gamma'\gamma\beta'\beta & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is this? Not symmetric, not pretty.
Admixture of a “boost” (velocity-change Lorentz transformation) and a rotation (which is also a kind of Lorentz transform, just a boring one)

If I just hand you a matrix, like this last one, how do you know if it’s a legit Lorentz transformation or not?

Again, it must preserve an invariant length ...

12: Invariant length in Relativity

TECHNISCHE UNIVERSITÄT DARMSTADT

The invariant length-squared of $x^\mu = (ct, x, y, z)$ is:

$$x^2 \equiv (ct)^2 - x^2 - y^2 - z^2 = (ct)^2 - \vec{x}^2$$

Note the minus sign.

- $x^2 > 0$: x^μ is timelike
- $x^2 < 0$: x^μ is spacelike
- $x^2 = 0$: x^μ is lightlike

$$x^2 = \begin{bmatrix} ct & -x & -y & -z \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

The row vector we need has the space entries sign-flipped. Define

$$x_\mu = g_{\mu\nu} x^\nu \, , \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad x^2 = x_\mu x^\mu$$

13: Invariant length and Metric

A Lorentz transform $\Lambda^{\mu}{}_{\nu}$ is any[2] transformation which preserves invariant length

$$
\begin{aligned}
x^2 \equiv x^{\mu} g_{\mu\nu} x^{\nu} &= x'^{\mu} g_{\mu\nu} x'^{\nu} \\
x^{\mu} g_{\mu\nu} x^{\nu} &= \Lambda^{\mu}{}_{\alpha} x^{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} x^{\beta} \\
x^{\mu} g_{\mu\nu} x^{\nu} &= x^{\alpha} \left(\Lambda_{\alpha}^{\top\,\mu} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} \right) x^{\beta} \\
x^{\mu} \left(g_{\mu\nu} \right) x^{\nu} &= x^{\mu} \left(\Lambda_{\mu}^{\top\,\alpha} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} \right) x^{\nu} \qquad\qquad g_{\mu\nu} = \Lambda_{\mu}^{\top\,\alpha} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu}
\end{aligned}
$$

or $\Lambda^{\top} g \Lambda = g$. Just like rotations but $\delta_{ij} \to g_{\mu\nu}$
Because $g_{\mu\nu}$ has one positive and 3 negative entries, we call these $\Lambda \in SO(3,1)$

[2] Also $\Lambda^{0}{}_{0} \geq 1$ so we don't reverse time, and Det $\Lambda = 1$ so we don't mirror-image space.

14: Tensors and Terminology

We call upper indices *contravariant*: x^μ, P^μ are *contravariant vectors*.
Lower indices are *covariant*: x_μ, P_μ are *covariant vectors*.
You can think of them as columns and rows, and $\Lambda^\mu{}_\nu$ as a matrix, *as long as* there are only a few indices.
When there are more indices, there is no good vector/matrix interpretation. But upper and lower indices transform under Lorentz transformations as:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \qquad x_\mu \to \Lambda_\mu{}^\alpha x_\alpha \text{ with } \Lambda_\mu{}^\alpha = g_{\mu\beta}\Lambda^\beta{}_\kappa g^{\kappa\alpha}$$

$$T^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} \qquad X_\mu{}^{\alpha\beta\gamma} \to \Lambda_\mu{}^\kappa \Lambda^\alpha{}_\sigma \Lambda^\beta{}_\delta \Lambda^\gamma{}_\tau X_\kappa{}^{\sigma\delta\tau}$$

Contravariant vector, covariant vector, rank-2 contravariant tensor, rank-4 tensor with one covariant and three contravariant indices.
Here $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$, which turns out to be the same.

$$g_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta g_{\alpha\beta} = g_{\mu\nu} \quad \text{is invariant. Similarly,} \quad T^\mu{}_\mu \to T^\mu{}_\mu$$

15: One more invariant

There is one more invariant: the totally antisymmetric tensor

$$\epsilon^{\mu\nu\alpha\beta} = -\epsilon^{\nu\mu\alpha\beta} = -\epsilon^{\alpha\nu\mu\beta} = -\epsilon^{\beta\nu\alpha\mu}\,,\ \epsilon^{0123} = 1$$

Careful: $\epsilon_{0123} = g_{00}g_{11}g_{22}g_{33}\epsilon^{0123} = -1$
Under a Lorentz transform

$$\epsilon^{\mu\nu\alpha\beta} \to \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\rho}\Lambda^{\alpha}{}_{\sigma}\Lambda^{\beta}{}_{\tau}\epsilon^{\kappa\rho\sigma\tau} = \epsilon^{\mu\nu\alpha\beta}\,\mathrm{Det}\,\Lambda$$

We know $g = \Lambda^{\top} g \Lambda$ so

$$\mathrm{Det}\,g = \mathrm{Det}\,\Lambda^{\top} g \Lambda = \mathrm{Det}\,\Lambda^{\top}\,\mathrm{Det}\,g\,\mathrm{Det}\,\Lambda = \mathrm{Det}\,g(\mathrm{Det}\,\Lambda)^2$$

so $\mathrm{Det}\,\Lambda = \pm 1$. If it's -1, ϵ changes sign – it's a pseudotensor.

16: Is there more to learn?

Oh yes! We haven't touched on how rotations and boosts fail to commute with each other, that is, on the group structure of $SO(3, 1)$ and what it implies. But I will leave that for a Quantum Field Theory course.

Next time we will talk about the consequences for energy, momentum, and what particles can do....