Teilchenphysik:

Lecture 21: Field theory in More Detail

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We want to understand the Higgs mechanism and the Standard Model.
To get there we have to think harder about how gauge theories work
And what happens when a gauge theory contains scalar fields

- Field theory in more detail
- Gauge fields in more detail
- Scalar fields and gauge fields
- Spontaneous symmetry breaking
- Higgs mechanism

Physics is an attempt to apply some physical laws to either

1. Predict how initial conditions (Q_i, P_i) will evolve, or
2. Explain how, if you start at Q_i and end at Q_f, you got from one to the other

If you can do one of these things, you can do the other.

Newton's laws set up to do 1.
Lagrange/Hamilton approach does 2.

3: How do we solve all physics

Action principle: All physics follows from the action S
The action is the time-integral of a Lagrangian L

$$S = \int_{t_i}^{t_f} dt \; L(Q(t), Q'(t))$$

If I know L and boundary (P_i, Q_i or Q_i, Q_f) data,
Then you can determine all dynamics.

- Classical physics: $Q(t)$ are values which extremize S
- Quantum physics: amplitude to go from $\langle \psi_i | Q_i \rangle = \psi_i(Q)$ at $t = t_i$ to $\langle \psi_f | Q_f \rangle = \psi_f(Q)$ at $t = t_f$ is

$$A(\psi_i, \psi_f) = \int_{\psi_i}^{\psi_f} \mathcal{D}Q \; \exp(iS(Q)/\hbar)$$

Path integral will be dominated by saddlepoints in $S(Q)$ if $\hbar$ is “small.” Saddlepoint condition is $\partial S/\partial Q = 0$ which says that small $\hbar$ limit of QM is classical mechanics.

4: Relativistic classical particles

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The action has to respect all symmetries.
That means S has to be the same in all reference frames.
Given trajectory $\vec{x}(t)$, what is the same in all frames?

- Rewrite $\vec{x}(t) \to x^\mu(\tau)$ with τ the proper time, $d\tau = dt/\sqrt{1 - v^2}$
- Total proper time is

$$\tau = \int_{x_i^\mu}^{x_f^\mu} d\tau = \int_{x_i^\mu}^{x_f^\mu} \sqrt{c^2 - v^2} dt = \int \sqrt{c^2 dt^2 - dx^2}$$

- Because $c^2 dt^2 - dx^2$ is the same in all frames, so is τ!
- Action is $S = mc\tau = mc \int_{x_i}^{x_f} d\tau$ so $L = mc^2 \sqrt{1 - v^2/c^2}$

$$\text{Canonical momentum:} \quad p_i = \frac{\partial L}{\partial v_i} = mv/\sqrt{1 - v^2/c^2}$$

$$\text{Hamiltonian:} \quad H = p_i v_i - L(x, p) = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

5: Relativistic field?

Suppose there is a field $\phi(x, t)$ varying through space and time.

$$S = \int dt\ L(\phi(x,t), \dot{\phi}(x,t))$$

Here L is a function of ϕ at *all* coordinates x.
Locality: It must be a *single space integral* over $\phi(x)$ and its derivatives:

$$L(\phi(x,t)) = \int d^3x\ \mathcal{L}(\phi, \partial_\mu \phi)$$

Lagrangian density $\mathcal{L}$ is function of ϕ at that point, and its space and time derivatives.

What should I expect the function $\mathcal{L}$ to look like?

6: Constraints on Lagrangian

The Lagrange density can't be anything you want. Rules?

- Depends on $\phi(x)$ and its derivatives – not on ϕ at other points (locality, causality) Without this, physics cannot be predictive.
- $\mathcal{L}$ should be spacetime scalar (Lorentz invariance)
- $\mathcal{L}$ must be Hermitian (so evolution is unitary)
- $\mathcal{L}$ should not contain too many powers of field+derivative

  This is totally not obvious. It's a deep Quantum Field Theory thing called renormalizability.
- $\mathcal{L}$ should be positive in Euclidean signature (stability)

  This makes sure Hamiltonian is bounded from below. Theories without this have dynamics which "blows up"

Real scalar field: not many things available:

$$\mathcal{L}(\phi) = \frac{Z}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \qquad V(\phi) = V_0 + V_1\phi + \frac{m^2}{2}\phi^2 + \frac{g}{6}\phi^3 + \frac{\lambda}{24}\phi^4$$

7: Interesting, puzzling case

Consider two-component scalar field ϕ_a with $a = 1, 2$
Suppose there is an $SO(2)$ symmetry rotating between components.
I will use summation conventions on the a indices.

$$\mathcal{L}(\phi_a) = \partial_\mu \phi_a \partial^\mu \phi_a - V_0 - \frac{m^2}{2}\phi_a\phi_a - \frac{\lambda}{24}\phi_a\phi_a\phi_b\phi_b$$

For $m^2 > 0$ the lowest-energy classical state has $\phi_1 = 0 = \phi_2$
The quantum vacuum is some sort of (small) fluctuations about this classical state.

But what if $m^2 = -\mu^2 < 0$? Theory is still stable, but ...

8: Spontaneous symmetry breaking

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My theory has a symmetry in which we rotate between ϕ_1, ϕ_2.

Potential energy is invariant:

$$\begin{aligned} V(\phi) &= V_0 - \frac{\mu^2}{2}\phi_a\phi_a + \frac{\lambda}{24}\phi_a\phi_a\phi_b\phi_b \\ &= K_0 + \frac{\lambda}{24}(\phi_a\phi_a - v^2)^2\,, \\ v^2 &= 6\mu^2 \end{aligned}$$

$V(\phi_a)$ minimized when $\phi_a\phi_a = v^2$
Not when $\phi = 0$.
There are many equally good vacua!

9: Goldstone's Theorem

My *theory* has a symmetry, but the *vacuum* does not.

Goldstone proved that, when this happens *with a global internal* symmetry, there are always massless particles, *Goldstone bosons,* associated with vacuum value varying through space.

Locally, field "feels" like it's vacuum. Only by comparing with neighboring regions – through gradients – can it realize there is a fluctuation occurring.

10: Scalar with global $SU(2)$ symmetry

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Suppose we have a *column vector* of complex scalars:

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + V(\phi^\dagger \phi) = \begin{bmatrix} \partial_\mu \phi_1^* & \partial_\mu \phi_2^* \end{bmatrix} \begin{bmatrix} \partial^\mu \phi_1 \\ \partial^\mu \phi_2 \end{bmatrix} + ...$$

If I apply $\phi \to U\phi$ with $U \in SU(2)$ a (constant) matrix, then the Lagrange density remains unchanged!

$$\phi \to U\phi, \quad \phi^\dagger \to \phi^\dagger U^\dagger\,, \quad \phi^\dagger \phi \to \phi^\dagger U^\dagger U \phi\,, \quad \partial_\mu \phi^\dagger \partial^\mu \phi \to \partial_\mu \phi^\dagger U^\dagger U \partial^\mu \phi$$

Essential that U be space-independent so I could move U past ∂^μ.

But can I expand symmetry so that $U \to U(x^\mu)$???

11: Gauge theory: the Problem

No you cannot!

$$\phi \to U\phi \quad \Rightarrow \quad \partial_\mu \phi \to \partial_\mu U \phi = U \partial_\mu \phi + \left(\partial_\mu U\right) \phi$$

Intuitively clear:

(Note, sufficient to consider $U = \mathbf{1} + i\lambda_a \tau_a/2$ with λ_a infinitesimal.
$\partial_\mu \phi \to (1 + i\lambda_a \tau_a/2)\phi + i\tau_a/2\, \phi \partial_\mu \lambda_a$

12: Gauge theory: the Solution

Derivative is a *comparator* between nearby points.
Add an $SU(2)$ rotation to that comparison!

$$\partial_\mu \phi \to \left(\partial_\mu - i\frac{\tau_a}{2} W^a_\mu\right)\phi \equiv D_\mu \phi$$

$$\partial_\mu \phi^* \to \left(\partial_\mu + i\frac{\tau^*_a}{2} W^a_\mu\right)\phi$$

$$\partial_\mu \phi^\dagger \to \phi^\dagger \left(\overleftarrow{\partial}_\mu + i\frac{\tau_a}{2} W^a_\mu\right) \equiv D_\mu \phi^\dagger = (D_\mu \phi)^\dagger$$

W^a_μ are *instructions* to rotate a little while you are comparing between nearby points.
Rotate ϕ at one point and not another: change how I rotate as I am comparing those points, and I can “undo the damage” of ϕ-rotation

13: Let's see if it works!

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I will apply a transformation:

$$\phi(x) \to \left(1 + i\frac{\tau_a}{2}\lambda_a(x)\right)\phi(x) \equiv U(x)\phi(x)\,, \qquad \text{and} \qquad W_a^\mu(x) \to W_a^\mu(x) + \delta W_a^\mu(x)$$

What I need is to ensure that $D^\mu\phi \to UD^\mu\phi$.
That way, $(D_\mu\phi)^\dagger D^\mu\phi \to (D_\mu\phi)^\dagger U^\dagger U D^\mu\phi$ is unchanged.
Let's see if there is *any* δW_a^μ for which this works!

$$D^\mu\phi = \left(\partial^\mu - i\frac{\tau_a}{2}W_a^\mu\right)\phi \to \left(\partial^\mu - i\frac{\tau_a}{2}W_a^\mu - \frac{\tau_a}{2}\delta W_a^\mu\right)\left(1 + i\frac{\tau_b}{2}\lambda_b\right)\phi$$

$$= \text{(original)} + (\text{Terms with } \lambda_b) + (-i\delta W_a^\mu \tau_a/2)\phi + \cancel{\mathcal{O}(\lambda\delta W)}$$

$$\text{Terms with } \lambda_b = i\frac{\tau_b}{2}\lambda_b D^\mu\phi + \left(i\frac{\tau_b}{2}\partial^\mu\lambda_b + \frac{[\tau_a\,,\ \tau_b]}{4}W_a^\mu\lambda_b\right)\phi$$

$$= i\frac{\tau_b}{2}\lambda_b D^\mu\phi + i\frac{\tau_c}{2}\left(\partial^\mu\lambda_c + \epsilon_{abc}W_a^\mu\lambda_b\right)\phi$$

$$\text{original} + \text{With } \lambda_b = \left(1 + i\frac{\tau_b}{2}\lambda_b\right)D^\mu\phi + i\frac{\tau_c}{2}\left(\partial^\mu\lambda_c + \epsilon_{abc}W_a^\mu\lambda_b\right)\phi$$

$$\text{Therefore, to cancel, } \frac{\tau_c}{2}\delta W_c^\mu = \frac{\tau_c}{2}\left(\partial^\mu\lambda_c + \epsilon_{abc}W_a^\mu\lambda_b\right) = D^\mu\lambda_c$$

14: Gauge transformation

Physics stays the same, including (covariant) derivatives, under transform

$$\phi(x) \to \left(1 + i\frac{\tau_a}{2}\lambda_a\right)\phi\,, \qquad W_a^\mu \to W_a^\mu + \partial^\mu\lambda_a + \epsilon_{abc}W_b^\mu\lambda_c \equiv W_a^\mu + D^\mu\lambda_a$$

But Lagrangian can also contain terms with *only* W, no ϕ.
What are allowed terms?

$W_a^\mu W_\mu^a \to W_a^\mu W_\mu^a + 2W_\mu^a D^\mu\lambda_a$ is *not* invariant!

$F_a^{\mu\nu} \equiv \partial^\mu W_a^\nu - \partial^\nu W_a^\mu + \epsilon_{abc}W_b^\mu W_c^\nu \to F_a^{\mu\nu} + \epsilon_{abc}F_b^{\mu\nu}\lambda_c$ At least *covariant*

$F_a^{\mu\nu}F_{\mu\nu}^a \to F_a^{\mu\nu}F_{\mu\nu}^a + \epsilon_{abc}F_b^{\mu\nu}F_{\mu\nu}^a\lambda_c + \epsilon_{abc}F_a^{\mu\nu}F_{\mu\nu}^b\lambda_c$ *Is* invariant, *and* a scalar!

Most general Lagrangian:

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - V(\phi^\dagger\phi) - \frac{1}{4g^2}F_{\mu\nu}^a F_a^{\mu\nu}$$

15: Physical interpretation of $F_a^{\mu\nu}$

16: Canonical normalization

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We originally said Lagrangian could be $ZD^{\mu}\phi D_{\mu}\phi + ...$
But then we stopped writing Z. Why? $\sqrt{Z}\phi_{\text{old}} = \phi_{\text{new}}$
We can get rid of $1/g^2$ factor by doing the same with W^{μ}_{a}: $\frac{1}{g}W^{\mu}_{a,\text{old}} = W^{\mu}_{a,\text{new}}$

$$\mathcal{L} = \frac{1}{4g^2}F^{\mu\nu}_{a,\text{old}}F_{\mu\nu a,\text{old}} = \frac{1}{4}F^{\mu\nu}_{a,\text{new}}F_{\mu\nu a,\text{new}}$$

$$F^{\mu\nu}_{a,\text{new}} = \partial^{\mu}W^{\nu}_{a,\text{new}} - \partial^{\nu}W^{\mu}_{a,\text{new}} + g\epsilon_{abc}W^{\mu}_{b,\text{new}}W^{\nu}_{c,\text{new}}$$

$$D^{\mu}\phi = \left(\partial^{\mu} - i\frac{\tau_a}{2}W^{\mu}_{a,\text{old}}\right)\phi = \left(\partial^{\mu} - ig\frac{\tau_a}{2}W^{\mu}_{a,\text{new}}\right)\phi$$

This is called **canonical normalization**.
A factor of g appears wherever W gives rise to nonlinearity.

17: What are W couplings doing?

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Look at the terms linear in W:

$$\phi^\dagger W_a^\mu \tau_a \phi = \begin{bmatrix} \phi_1^* & \phi_2^* \end{bmatrix} \begin{bmatrix} W_3^\mu & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & -W_3^\mu \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$= 2W_+^\mu \phi_2^* \phi_1 + 2W_-^\mu \phi_1^* \phi_2 + W_3^\mu (\phi_1^* \phi_1 - \phi_2^* \phi_2)$$

The $W_\pm$ mix upper and lower components.
The W_3 couples to upper and lower with opposite strengths.
Just like the left-handed part of $W^\pm$ and Z couplings!

18: What have we learned so far?

- Physics is determined by action.
- Field theory: Action is spacetime integral of Lagrange density
- Lagrange density must be scalar function of fields
- Scalar fields are a real possibility
- Scalar fields can have strange spontaneous symmetry breaking effect
- Gauge theory expands notion of symmetry to allow spacetime-dependent transformations. Cost: introduction of a comparator W_a^μ
- For nonabelian symmetry like $SU(2)$, W have more complex transformation law.
- W enter action through squared field-strength. Nonlinear
- A mass term for gauge fields is *not allowed!* **Massless!**

But what happens when *Spontaneous symmetry breaking* meets *Nonabelian gauge theory*? Find out on Friday!