Teilchenphysik:

Lecture 3: Kinematics

TECHNISCHE UNIVERSITÄT DARMSTADT

Kinematics: understanding what Relativity says about energy and momentum for particles and collisions

- Proper time, proper velocity
- Energy+momentum as a 4-vector
- Collisions: elastic vs inelastic
- Different frames: what's the same, what's different, and why do we collide beams instead of fixed-target?

2: Proper time

Proper time is the time YOU (or a particle, or...) observe to go by.
If you are moving, the true time is $t = \gamma\tau$ or

$$\frac{dt}{d\tau} = \gamma$$

Similarly, velocity is $\vec{v} = \frac{d\vec{x}}{dt}$ but **proper velocity** is $\vec{\eta} = \frac{d\vec{x}}{d\tau}$.
The elapsed proper time is the same in all frames!
Since x^μ transforms as a 4-vector, so does

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} = \begin{bmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{bmatrix} \quad \text{and} \quad \eta_\mu \eta^\mu = c^2 \tag{1}$$

or 1 in civilized units. Call η^μ the **proper 4-velocity**

3: Energy and momentum

The book's approach is to claim that, for momentum to make sense within relativity, it has to transform in a way which is consistent relativistically, and this is

$$\vec{p} = m\vec{\eta}$$

We can name the 0 or time-element of η^μ as E/c to write:

$$p^\mu = m\eta^\mu\,, \qquad p^0 = E/c, \quad p^i = \vec{p}$$

One then finds that $p_\mu p^\mu = E^2/c^2 - \vec{p}^2 = m^2c^2$ is invariant.

I think about things differently. What is the *definition* of energy and of momentum?
Given a particle with wave function $\psi(t, x) = e^{i(\vec{k}\cdot\vec{x} - \omega t)}$,

- energy is how fast the wave function varies in time:

$$E\psi = i\hbar \frac{\partial}{\partial t}\psi = +\hbar\omega\psi$$

- Momentum is how fast the wave function varies in space:

$$\vec{P}\psi = -i\hbar \frac{\partial}{\partial \vec{x}}\psi = +\hbar\vec{k}\psi$$

Note the relative - sign, which you learned in your QM class!
This sign is what makes sure that the wave is moving forward.

5: Wave functions and relativity

TECHNISCHE UNIVERSITÄT DARMSTADT

How does a wave function change in relativity? *The value of the wave function at a spacetime event is the same for all observers* $\psi(x^\mu) \to \psi(\Lambda^\mu{}_\nu x^\nu)$ under Lorentz transformation.

Frame: particle at rest

Frame: particle velocity v

6: Wave function and relativity

Wave function of particle at rest, in the primed coordinates:

$$\psi(x'^{\mu}) = e^{-i(mc/\hbar)ct'} = e^{-i(mc/\hbar)x'^0}$$

Lorentz transformation to a frame where particle has $\beta = v/c$:
New frame $x^0 = \gamma x'^0 + \beta\gamma x'^1$ and $x^1 = \gamma x'^1 + \beta\gamma x'^0$ or $x'^0 = \gamma(x^0 - \beta x^1)$

Wave function: $\psi(x^\mu) = e^{-i(mc/\hbar)(\gamma ct - \beta\gamma x^1)}$

$$\text{Energy } E\psi = i\hbar\frac{\partial}{\partial t}e^{-i(mc/\hbar)(\gamma ct-\beta\gamma x^1)} = \gamma mc^2\,\psi \qquad \checkmark$$

$$\text{x-momentum } p^1\psi = -i\hbar\frac{\partial}{\partial x^1}e^{-i(mc/\hbar)(\gamma ct-\beta\gamma x^1)} = \beta^1\gamma mc\,\psi = \gamma m v_x\,\psi \qquad \checkmark$$

Note: we can combine derivatives as *covariant* 4-vector:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,, \qquad p^\mu\psi = i\hbar g^{\mu\nu}\partial_\nu\psi\,.$$

7: Nonrelativistic, ultrarelativistic limits

TECHNISCHE UNIVERSITÄT DARMSTADT

For small $\beta \ll 1$ we can expand:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = mc^2 + \frac{mv^2}{2} + \frac{3mv^4}{8c^2} + ...$$

$$\vec{p} = \gamma m\vec{v} = m\vec{v} + m\vec{v}\frac{v^2}{2c^2} + ...$$

For $\beta \to 1$ we can treat γ as large and expand β about 1:

$$\gamma^2 = \frac{1}{1 - \beta^2} \quad \Rightarrow \quad \beta = \sqrt{1 - \gamma^{-2}} \simeq 1 - \frac{1}{2\gamma^2} + ...$$

Energy is $E = \gamma mc^2$, momentum is $pc \simeq E(1 - 1/2\gamma^2) = E - m^2/2E$
Energy and momentum are almost equal.
This is *a common limit* in particle physics!

8: Massless particle

We can get to "massless particle" by taking $m \to 0$ limit.
Careful:

- Limit $m \to 0$ at *fixed* v means that $E = \gamma mc^2 \to 0$ and $p = \gamma mv \to 0$
- Limit $m \to 0$ at *fixed* E makes more sense, corresponds to a physical state with finite energy but no mass.

Limit $m \to 0$ at fixed E: $|\vec{p}| = E$, and $|\vec{v}| = 1$.
Proper time and η^μ no longer make sense.
But I can define η^μ / m and its limit makes sense.

A massless particle has no rest frame.

9: Two particles?

What if I have two particles?
Particle 1: energy E_1,
momentum $\vec{p}_1$
Particle 2: energy E_2,
momentum $\vec{p}_2$
They get close to each other:
No idea what happens.

What are properties of the *combined system?*

10: Collision possibilities

TECHNISCHE UNIVERSITÄT DARMSTADT

Particles, far apart, have $E = E_1 + E_2$ and $\vec{p} = \vec{p}_1 + \vec{p}_2$.
When particles get close, they interact. Interaction energies, momenta.
When “fragments” or “daughters” escape, we again have the sum of the parts.

- Two particles can fuse into one.
- They can scatter into two of the same sort of particle
- They can scatter into two different particles
- They can turn into 3 or more particles

What are available energies and momenta in each case?

We know final $E = E_1 + E_2$ and $\vec{p} = \vec{p}_1 + \vec{p}_2$ by energy & momentum conservation.

11: Rest mass of the system

Consider first that the particles merge into one particle.
What is its mass?

$$M^2c^4 = E^2 - p^2c^2 = c^2p^\mu p_\mu$$

But $p^\mu = p_1^\mu + p_2^\mu$. So

$$\begin{aligned} M^2c^2 &= (p_1 + p_2)^\mu(p_1 + p_2)_\mu = (E_1 + E_2)^2/c^2 - (\vec{p}_1 + \vec{p}_2)^2 \\ &= p_1^\mu p_{1\mu} + p_2^\mu p_{2\mu} + 2p_1^\mu p_{2\mu} = m_1^2c^2 + m_2^2c^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2) \end{aligned}$$

This is an invariant, called the system mass-squared.

There is a unique frame, the **center-of-mass frame**, where $\vec{p}_1 = -\vec{p}_2$.
In this frame the final particle is at rest, and $Mc^2 = E_1 + E_2$.
In all other frames, $Mc^2 < E_1 + E_2$.

12: Fixed target vs beam-on-beam

TECHNISCHE UNIVERSITÄT DARMSTADT

Suppose an accelerator can bring a proton, mass $m \simeq 1\,\text{GeV}$, to 100 GeV energy. How heavy is the heaviest particle I can make if I collide it with a proton at rest?

$$
\begin{aligned}
p^\mu &= mc\Big[(100+1)\ (99.995+0)\ 0\ 0\Big] \\
p^\mu p_\mu &= m^2c^2(10201-9999) = 202m^2c^2 = M^2c^2 \\
M &= m_p\sqrt{202} = 14.2\,\text{GeV}
\end{aligned}
$$

What if I collide two protons with that energy, flying head-to-head?

$$
\begin{aligned}
p^\mu &= mc\Big[(100+100)\ (99.995-99.995)\ 0\ 0\Big] \\
p^\mu p_\mu &= m^2c^2(40000-0) \\
M &= 200\,\text{GeV}
\end{aligned}
$$

13: 2 body scattering

TECHNISCHE UNIVERSITÄT DARMSTADT

What if 2 particles come out?
One possibility – **elastic scattering** – is that they have the same masses as the incoming particles. Indeed, they are often *the same species* as the incoming particles.
In the Center of Mass frame, this looks like

$$p_1^\mu = \begin{bmatrix} \gamma_1 m_1 c \\ \beta_1\gamma_1 m_1 c \\ 0 \\ 0 \end{bmatrix} \qquad p_2^\mu = \begin{bmatrix} \gamma_2 m_2 c \\ -\beta_2\gamma_2 m_2 c \\ 0 \\ 0 \end{bmatrix}$$

$$k_1^\mu = \begin{bmatrix} \gamma_1 m_1 c \\ \beta_1\gamma_1 m_1 c\cos\theta \\ \beta_1\gamma_1 m_1 c\sin\theta \\ 0 \end{bmatrix} \qquad k_2^\mu = \begin{bmatrix} \gamma_2 m_2 c \\ -\beta_2\gamma_2 m_2 c\cos\theta \\ -\beta_2\gamma_2 m_2 c\sin\theta \\ 0 \end{bmatrix}$$

where $\beta_1\gamma_1 m_1 = \beta_2\gamma_2 m_2$ and θ is the **scattering angle**.
Here k_1, k_2 are the final momenta.

14: Inelastic scattering

Can the final masses be different than the initial masses?
You bet!

- The final masses are required to obey
  $m_1 c + m_2 c \leq \sqrt{(p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu})} = Mc$
  so there is enough *available* energy to create the particles
- Same as saying that $m_1 c^2 + m_2 c^2 < E_{\text{tot}}$ in the CM frame
- Any smaller final masses are allowed.
- $m_{1,\text{init}} + m_{2,\text{init}} > m_{1,\text{fin}} + m_{2,\text{fin}}$: "explosive"
- $m_{1,\text{init}} + m_{2,\text{init}} = m_{1,\text{fin}} + m_{2,\text{fin}}$: "elastic"
- $m_{1,\text{init}} + m_{2,\text{init}} < m_{1,\text{fin}} + m_{2,\text{fin}}$: "sticky" or lossy

Nobody ever said that mass is conserved. In fact, it generally isn't.

15: Example: equal mass

Two particles collide. The system rest mass is M.
Two particles, mass m, emerge. What are their velocities?

In the center of mass frame, each will carry away half the energy.

we have
$$E = \frac{Mc^2}{2} = \frac{mc^2}{\sqrt{1-\beta^2}}$$

Therefore
$$1-\beta^2 = \frac{4m^2}{M^2}$$
$$\beta = \sqrt{1-\frac{4m^2}{M^2}}$$

Only possible if $m \leq M/2$. For $m/M \ll 1$, $\beta \to 1$.

16: Example: betatron

TECHNISCHE
UNIVERSITÄT
DARMSTADT

The betatron performed $p + p \to p + p + p + \bar{p}$ where p is a proton and $\bar{p}$ is the antiproton, which has the same mass.
One p is moving, the other is at rest.
What energy must the moving p have?

To produce the desired final state we need $M \geq 4m_p$.

$$16m_p^2 = (p_1 + p_2)^\mu (p_1 + p_2)_\mu = p_1^2 + p_2^2 + 2p_1^\mu p_{2\mu} = 2m_p^2 + 2E_1 m_p/c^2$$

Therefore $2E_1 m_p/c^2 = 14m_p^2$ or $E_1 = 7m_p c^2$.
The proton needs a γ-factor of 7, that is, we need to *add* 6 times its rest-mass in energy.

If we do this in a beam-on-beam experiment, each p needs only have its energy doubled, that is, $m_p c^2$ energy added to each.