Teilchenphysik:

Lecture 4: Symmetry I

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Another preliminary topic before we discuss actual particles
Symmetries help *organize* the theory, so we need to know about them first. You know parts of this story

- What is a symmetry: transformation on space/theory/something which leaves it unchanged
- How do symmetries fit together: mathematical groups
- What do symmetries give us: conservation laws etc
- Important example: angular momentum and spin
- Important example: internal rotations, isospin ...

I don’t know if we will reach isospin today but we can try.

2: Symmetry at work: my favorite example

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I will introduce you to some strongly-interacting (meson) particles:

- "The" pion or pi-meson: really three particles.
  π^+, π^0, π^- with masses 139.57 MeV, 134.98 MeV, 139.57 MeV
- The η meson: η^0, mass 547.86 MeV
- "The" ρ meson: really three particles.
  ρ^+, ρ^0, ρ^- all with $m \simeq 775$ MeV

The pions are the lightest strong-interaction particles. They can only decay weakly ($\pi^\pm$) or through EM (π^0), which is slow.

The other two have enough energy to decay to pions, which is faster.

- The ρ meson decays with a lifetime of $\tau = \frac{1}{149\text{ MeV}}$
- The η decays with a lifetime of $\tau = 1/0.00131$ MeV

Why is the η 100,000 times longer lived???

3: Explanation: Parity (a symmetry)

The strong interactions respect a symmetry called *parity*.
Parity is “reflection in a mirror.”
When you reflect in a mirror, Nature stays the same *if*

- you leave “parity-even” particles’ wave functions the same
- you multiply “parity-odd” particles’ wave functions by -1

You saw this with Harmonic Oscillator states in QM.

How do our particles behave under parity?

- π^{+0-} are parity-odd and spin 0
- η^0 is parity-odd and spin 0
- ρ^{+0-} are parity-odd *but spin 1*

Parity conservation: $\eta \to \pi\pi$ is *forbidden.*
$\eta \to \pi\pi\pi$ is complex, has barely enough energy ...
But $\rho^+ \to \pi^+\pi^0$ is *allowed* (L = 1 ...)

4: What are symmetries good for?

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- They organize the states (particles) and fields of the theory.
- They give *exact statements* about what processes can and cannot occur
- Some (continuous) symmetries lead to *conservation laws*
- Even if a symmetry is not exact, it can explain *patterns*
  and it can give *good approximate* results which are often systematically improvable

Therefore we will spend some time understanding them *before* looking at the particle theories themselves.

5: Symmetries as groups

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A symmetry involves applying a *transformation*

- on space and/or time (rotation, parity, Lorentz, etc): **spacetime symmetry**
- on particles themselves or (equivalently) the fields which describe them (EM field, other fields we haven't met yet): **internal symmetry**

Call the operation of such a transformation R. Basic ground rules:

- Identity: the transformation I where nothing changes is a symmetry.
- Product: I can apply one symmetry transformation R_1 and then apply another, R_2. That gives me a new transformation $R' = R_2 R_1$ which is also a symmetry.
- Inverse: For any transformation R, there is a transformation which *undoes* that transformation, R^{-1}: $R^{-1}R = I = RR^{-1}$
- Associativity: $(R_1 R_2)R_3 = R_1(R_2 R_3)$

That makes the set of symmetry transformations a *mathematical group*.

6: Simple example

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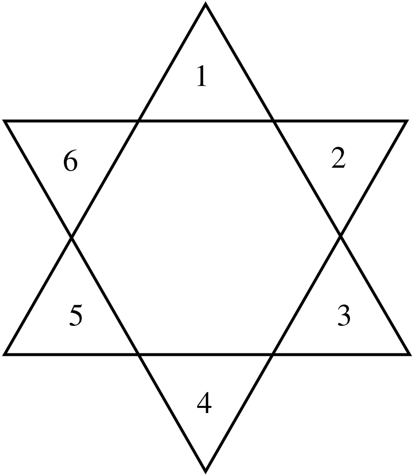

7: Different kinds of symmetries

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What does the symmetry act on?

- spacetime symmetry: $x^\mu \to \Lambda^\mu{}_\nu x^\nu + \xi^\mu$
- internal symmetry: flip matter $\leftrightarrow$ antimatter ...

How "big" is the symmetry group?

- Discrete symmetry groups (C, P, T, point groups)
- Continuous symmetry groups (phase rotation, space-rotation, etc)

Does nature *really* respect the symmetry?

- exact symmetries of nature (gauge, spacetime, CPT, B? $B - L$?)
- approximate but actually broken symmetries (isospin, P, C, T)

Just because a symmetry isn't exact doesn't mean it's useless.

8: Continuous symmetries

Continuous symmetries are controlled by continuous (Lie) symmetry groups.
These are our favorites because of **Nöther's Theorem**:
each "generator" of a continuous symmetry gives us a conserved current.

Spacetime symmetry involves the *Poincaré Group*, which combines translations with the (noncompact) group of Lorentz transforms $SO(3,1)$

The **Coleman-Mandula theorem** says that the group of symmetries is a simple product of spacetime symmetries $\times$ internal symmetries.

So what are the possible internal symmetries?

9: Possible internal symmetries

Continuous internal symmetries must be *compact Lie groups.*
There is a complete classification of all such groups:

- The group $U(1)$ of phase rotations $e^{i\theta}$
- The groups $SU(N)$ of **special unitary** transformations $N = 2, 3, ...$
- The groups $SO(2N)$ of **orthogonal** transformations for even N
- The groups $SO(2N + 1)$: same as above but odd N
- The groups $SP(N)$ of symplectic transformations (you never need)
- A few leftovers: G_2, F_4, E_6, E_7, E_8

and any product group built from these.

Beyond-standard-model theorists worry about many of these ($SU(5)$, $SO(10)$)
String theorists are obsessed with E_8 and $SO(32)$.

But the Standard Model only contains $U(1)$, $SU(2)$, and $SU(3)$.

10: Groups and their representations

Particles (or fields) don't generally transform *directly* under the symmetry group. They transform under a *representation* of the group.

Imagine there are a *finite* number of particle-types
(π^+, π^0, π^-) or (p, n) or (ρ^+, ρ^0, ρ^-) or (η^0)
The symmetry mixes them up with each other.
That requires *finite-dimensional* matrices.
Each group element is *represented* by a matrix; the product of two group elements is represented by the product of those matrices.

You have secretly already learned about this, as it's how spin works

11: Reminder: rotations

Rotation on a state $|\psi\rangle$ performed by operator $U(\theta_i)$.
generated by angular momentum operators: z-rotation: $U(\theta_z) = \exp(-i\theta_z \hat{J}_z)$.
Rotations don't commute, so $\hat{J}_i$ operators have *commutation relations*

commutator: $\left[\hat{J}_i\,,\,\hat{J}_j\right] = i\hbar\epsilon_{ijk}\hat{J}_k$

defining $\hat{J}_\pm \equiv \dfrac{\hat{J}_x \pm i\hat{J}_y}{2}$, states: $|j, m\rangle$ (definite $\hat{J}^2, \hat{J}_z$ values)

$$\hat{J}_z|j, m\rangle = \hbar m|j, m\rangle \qquad \hat{J}_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}|j, m \pm 1\rangle$$

As you know, this forces quantization:
$j \in \{0, 1/2, 1, 3/2, ...\}$ and $m \in \{j, j-1, ..., -j\}$.

12: Spin-$\frac{1}{2}$ or Fundamental Representation

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Simplest example is spin-$\frac{1}{2}$. Use $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$ as basis

$$|\psi\rangle = c_\uparrow \left|\frac{1}{2},\frac{1}{2}\right\rangle + c_\downarrow \left|\frac{1}{2},-\frac{1}{2}\right\rangle \equiv \begin{bmatrix} c_\uparrow \\ c_\downarrow \end{bmatrix}$$

$$\hat{J}_z = \hbar \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = \frac{\hbar}{2}\sigma_z$$

Pauli matrices

$$\hat{J}_x = \hbar \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_x$$

$$\hat{J}_y = \hbar \begin{bmatrix} 0 & \frac{-i}{2} \\ \frac{i}{2} & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_y$$

Rotation matrices are $U(\theta_i) = \exp(-i\theta_i\sigma_i/2)$. They also obey $[\hat{J}_i\,,\,\hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$.

These *matrix* expressions for $\hat{J}_i$ and $U(\theta_i)$ are called a **representation**

13: Spin-1 or adjoint representation

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Now my basis has 3 elements: $|\psi\rangle = c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle$

Write this as a column matrix: $$|\psi\rangle = \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix}$$

$$\hat{J}_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \hat{J}_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

The rotation operator is still $U(\theta_i) = \exp(-i\theta_i \hat{J}_i)$ but with these new matrix $\hat{J}$ values.
We can also check that $[\hat{J}_i\,,\ \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$.

14: Spin-0 or trivial representation!

For a spin 0 particle, the wave function has one component: $|\psi\rangle = c_0|0, 0\rangle$

$$\hat{J}_z = \hbar\,[0] \qquad \hat{J}_x = \hbar\,[0] \qquad \hat{J}_y = \hbar\,[0]$$

The rotation operator is $U(\theta_i) = \exp(-i\theta_i\hat{J}_i) = 1$
and $[\hat{J}_i\,,\,\hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$ is now trivially true!

This representation is “unfaithful.” Information about rotations is lost.

Actually, spin-1 loses a tiny detail as well: a 360° rotation is the same as 0°, whereas for spin-$\frac{1}{2}$ there is a -1 which makes them distinct.

Difference between $SU(2)$ and $SO(3)$ . . .

15: What's general?

- When I have a continuous symmetry, it has **generators**
- They obey some **commutation relations**
- States, or particles, form **multiplets**
- The generators act on these through matrices called **representation matrices**
- Rep. matrices obey the same commutation relations as Generators
- Trivial, "fundamental," and other (usually larger) reps

But what if I combine two particles which are each in some representation?

16: Products of representations

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We have seen this for spin. Spin j_1 and spin j_2 combine into states of generic form $|j_1, m_1\rangle\, |j_2, m_2\rangle$

$$\text{most generic state:} \quad \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} c_{m_1,m_2} |j_1, m_1\rangle\, |j_2, m_2\rangle$$

Each piece transforms according to J in that rep.
Combination is tensor product of the two reps. $(2j_1 + 1)(2j_2 + 1)$ dimensional.
But generally *reducible*

17: Reducibility for Spin

Product of spin-$5/2$ and spin-1 for instance:

$$\frac{5}{2} \otimes 1 = \frac{7}{2} \oplus \frac{5}{2} \oplus \frac{3}{2}$$

more generally, if $j_1 \geq j_2$,

$$j_1 \otimes j_2 = \bigoplus_{j=j_1-j_2}^{j_1+j_2} j$$

specifically

$$|j_1 m_1\rangle \, |j_2 m_2\rangle = \sum_j C^{jj_1 j_2}_{mm_1 m_2} |jm\rangle$$

with $C^{jj_1 j_2}_{mm_1 m_2}$ the Clebsch-Gordan coefficients (entries in rotation matrix from "natural" to "irreducible" basis)

You know all of this. But this is generic; the same happens for other Lie groups.

18: Summary

Symmetries:

- Transformations on states (spacetime, fields, particle types) which leave "physics" invariant
- Form mathematical groups
- Lead to exact results, conservation laws, organize theories
- Continuous symmetries have
  - group generators with commutation relations
  - states transforming under representations
  - product representations which are reducible

  with rotations / angular momentum being a good example you know