Teilchenphysik:

Lecture 6: Feynman Rules

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So what is it that we want to calculate?

- What does 1 particle do? Decay rates / widths and partial widths
- What do 2 particles do? Scattering cross-sections (total, differential)
- What else can 2+ particles do? Bound states (skip for now)

How do we calculate them?

- Amplitudes and matrix elements
- Fermi's Golden Rule
- Feynman calculus to find matrix elements

Today we talk about *defining* the things we want, and the *kinematics* of production

2: Decay rates

Imagine that I have an unstable particle A. What does that mean?

At time $t = 0$: $|\psi\rangle = |A\rangle$ but at time t_1: $e^{-iHt_1}|\psi\rangle = e^{-im_A t_1} c_1 |A\rangle + c_2 |B, C\rangle$

where $|B, C\rangle$ is a state with no A particle, but with some other stuff instead.
Probability to still be an A particle: $c_1^* c_1$.
Probability to be something else: $c_2^* c_2 = 1 - c_1^* c_1$

Suppose A is *relatively long-lived*: after time $t_1 \gg 1/m_A$, we still have $c_2^* c_2 \ll 1$, $c_1^* c_1 \simeq 1$. Then B, C particles tend to fly away and escape

3: Decay rate as exponential process

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If $|B,C\rangle$ flies away, it never evolves *back* into $|A\rangle$:

$$e^{-iHt_1}|A\rangle = e^{-im_At}c_1|A\rangle + c_2|B,C\rangle$$
$$e^{-2iHt_1}|A\rangle = e^{-2im_At}c_1^2|A\rangle + e^{-im_At}c_1c_2|B,C\rangle + c_2e^{-iHt}|B,C\rangle$$
$$e^{-3iHt_1}|A\rangle = e^{-3im_At}c_1^3|A\rangle + e^{-2im_At}c_1^2c_2|B,C\rangle + e^{-im_At}c_1c_2e^{-iHt_1}|B,C\rangle + c_2e^{-2iHt_1}|B,C\rangle$$

The *probability* to *still be* particle-type A goes, at times $(0, t_1, 2t_1, 3t_1, ...)$, as $(1, |c_1|^2, |c_1|^4, |c_1|^6, ...)$ which is a *geometric series*

Probability to still be A decays exponentially:
Prob(A at time t)= $e^{-\Gamma t} = e^{-t/\tau}$
We call Γ the *decay width* and τ the *decay time*

Strategy will be based on computing $c_2^* c_2$, not c_1.

4: Multiple final states

There are almost always multiple possible final states.

$$e^{-iHt}|K^+\rangle = e^{-im_K t}c_1|K^+\rangle + c_2|\mu^+\nu_\mu\rangle + c_3|\pi^+\pi^0\rangle + c_4|\pi^+\pi^+\pi^-\rangle + ...$$

Decay depends on $\frac{c_2^* c_2}{t} = \Gamma_{\mu^+\nu_\mu}$, $\frac{c_3^* c_3}{t} = \Gamma_{\pi^+\pi^0}$, $\frac{c_4^* c_4}{t} = \Gamma_{\pi^+\pi^+\pi^-}$, ...
We call these *partial widths* for each decay process.
The sum is the *total width* $\Gamma = \Gamma_1 + \Gamma_2 + ...$

We also introduce the *branching ratio*

$$\mathrm{Br}_i = \frac{\Gamma_i}{\Gamma} = \text{probability that final state will be } i$$

5: Next topic: Cross-section

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Question: I shoot something (electron? arrow?) at a target (proton? bull's eye?).
How likely am I to hit?
Depends on three things:

- How "big" is the target? **Cross section**
- How many times do I shoot?
- How good is my aim?
  Product of (times I shoot)$\times$(aim) = **Luminosity** (particles / cm^2 s)

Number of "hits" per second is Luminosity $\times$ Cross section.

Luminosity: how good an accelerator engineer do you have?

Cross-section: something for theorists to compute.

6: Classical example: hard sphere

Shoot small rubber balls at a hard sphere of radius R

Distance b from axis (**impact parameter**): scattering occurs if $b < R$.
Total cross-section $\sigma = \pi R^2$
Units of area

Angle scattered:
$\theta = 2\arccos(b/R)$
(and $\phi_{\text{out}} = \phi_{\text{in}}$)

$b = 0$: $\theta = \pi$. $b \to R$: $\theta \to 0$

7: Hard Sphere: differental cross-section

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We have more info than total cross-section $\sigma = \pi R^2$.
We can say *how many* balls get scattered into each angle θ, or really each *solid-angle* $\sin(\theta) d\theta d\phi$

Area of incident beam with $b \in [b_0, b_0 + db]$ and $\phi \in [\phi_0, \phi_0 + d\phi]$ is $b\, db\, d\phi$.

differential cross-section (note, $d\Omega = \sin(\theta)\, d\theta\, d\phi$)

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega \qquad \text{Hard sphere:} \quad \frac{d\sigma}{d\Omega} = \frac{b\, db\, d\phi}{\sin(\theta) d\theta d\phi} = \frac{b}{\sin(\theta)} \frac{db}{d\theta}$$

We already found $b = R\cos(\theta/2)$, $|db/d\theta| = -R\sin(\theta/2)/2$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)} \frac{db}{d\theta} = \frac{R\cos(\theta/2)}{\sin(\theta)} \frac{R\sin(\theta/2)}{2} = \frac{R^2}{4} \quad \text{isotropic!}$$

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It is important to understand that the strict geometrical picture from the hard-spheres example is deceptive.
For *large* objects (nuclei) and high enough energies, b is well defined and a geometrical picture is OK. But note:

$$\vec{b} \times \vec{p} = \vec{L} \quad \text{angular momentum} \quad \text{classical if} \quad bp \gg \hbar$$

Nuclear size $b \sim\sim 5\,\text{fm}$, classical for $p > 40$ MeV. Proton is $10\times$ smaller, need 400 MeV to see its physical size ($p < 400$MeV sees proton as point particle)

Otherwise, scattering really a quantum process.
fundamental particles have no geometrical size – always essentially quantum.

9: Fermi Golden Rule: Decays

Decays happen because of *interaction Hamiltonian* and *time-dependent perturbation theory*. Consider Hamiltonian:

$$H = \text{“}\partial_\mu A \partial^\mu A + m^2 A^2 + \text{ same for } B, C\text{”} + \text{“}ABC\text{”}$$

There are parts of H which “tell” particles to propagate, and what masses they have, and a part which describes interactions:

$$\langle A | H_{ABC} | BC \rangle \neq 0$$

If I have a state containing particle A, it constantly produces an amplitude to have BC instead.

Produces *all possible BC* combinations consistent with momentum conservation.

10: Possible decay final states

Consider an A particle at rest. What B, C are possible?

Any final momenta so long as they add up to p_A (0 in rest frame)

11: Which final state gets made?

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All of them! It's QM, they each have an *amplitude*

$$\langle BC|H|A\rangle \neq 0 \qquad\qquad \text{if } H_{ABC} \neq 0$$

$$H|A\rangle = c_1|B_{p_1}C_{-p_1}\rangle + c_2|B_{p_2}C_{-p_2}\rangle + ...$$

Decay into QM superposition of *all* B, C states.
BUT each B, C state still evolves with time!
Tiny bit of time-evolution: (lots of $1/\hbar$ left out!!)

$$e^{-iHdt}|A\rangle = e^{-iE_A dt}|A\rangle + c_1\,dt\,|BC_1\rangle + ...$$

$$e^{-iHdt}e^{-iHdt}|A\rangle = e^{-2iE_A dt}|A\rangle + c_1\,dt\,e^{-iE_A dt}|BC_1\rangle + c_1\,dt\,e^{-iE_{BC_1}dt}|BC_1\rangle + ...$$

First BC term: generated from $|A\rangle$.
Second BC term: time evolution from first time.
Longer time t_1: BC generated at every time, each picks up a different phase:

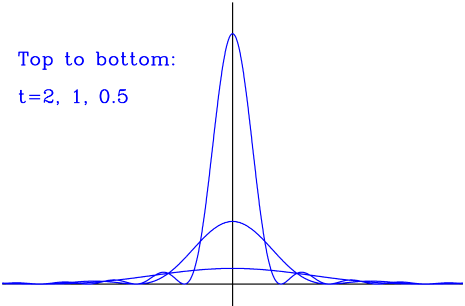
$$\text{At time } t_1: \quad \int_0^{t_1} dt\, c_1 e^{-iE_A t} e^{-iE_{BC_1}(t_1-t)}\, |BC_1\rangle$$

12: Energy selection

What if $E_{BC_1} \neq E_A$? Call $\Delta E = E_{BC_1} - E_A$.

Amplitude at time t: $\int_0^{t_1} dt\, c_1 e^{-iE_A t} e^{-iE_{BC_1}(t_1 - t)} = c_1 e^{-iE_A t} \int_0^t dt'\, e^{-it'\ \Delta E} dt'$

Squared amplitude: $\left| \int_0^t dt'\, e^{-it'\ \Delta E} dt' \right|^2 = \dfrac{4 \sin^2(\Delta E\, t/2)}{\Delta E^2}$



As t increases, function becomes more peaked. Total area under curve $= \pi t$ linear in time.
Late times: $E_{BC} = E_A$ or probability $\to 0$

13: Decay rate

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Taking t "large" ($mc^2 t/\hbar \gg 1$ so $\Delta E \ll mc^2$)
Decay rate is phase space $\times$ p^μ conservation $\times$ "Matrix Element"

$$\Gamma_{A\to BC} = \frac{1}{2m_A}\int \frac{d^3p_B\, d^3p_C}{(2\pi)^3 2E_B\, (2\pi)^3 2E_C} \times (2\pi)^4 \delta^4(p_A^\mu - p_B^\mu - p_C^\mu) \times |\mathcal{M}_{A\to BC}|^2$$

$$\Gamma_{A\to BCD} = \frac{1}{2m_A}\int \frac{d^3p_B\, d^3p_C\, d^3p_D}{(2\pi)^3 2E_B\, (2\pi)^3 2E_C\, (2\pi)^3 2E_D} \times (2\pi)^4 \delta^4(p_A^\mu - p_B^\mu - p_C^\mu - p_D^\mu) \times |\mathcal{M}_{A\to BCD}|^2$$

- Meaning of $|\mathcal{M}_{A\to BC}|^2$: squared amplitude for *microphysics* to turn A into BC. Roughly $|\langle BC|H|A\rangle|^2$, all info about theory is here
- Factor $1/2m_A \to 1/2E_A$ if particle not at rest. Accounts for time dilation
- Combination $d^3p/2E_p$ is the same in all rest frames.

14: And scattering?

When A and B scatter into C and D, the expression is similar.
Cross section is:

$$\sigma_{AB\to CD} = \frac{1}{2E_A\,2E_B\,|\vec{v}_A - \vec{v}_B|}\int \frac{d^3p_C\;d^3p_D}{(2\pi)^3 2E_C\;(2\pi)^3 2E_D} \times (2\pi)^4\delta^4(p_A^\mu + p_B^\mu - p_C^\mu - p_D^\mu) \times |\mathcal{M}_{AB\to CD}|^2$$

Each extra final particle has the same integration factor.

Why this leading factor? Dimensional grounds: σ, an area, should scale with E^{-2}.
And this combination is frame independent.

Intuition: $1/(v_A - v_B)$ “flux factor” means slow-moving particles,
with more time to interact, have larger cross-section.

15: Three-body scattering?

This is almost impossible to make happen in a particle physics experiment. Hard enough to get 2 bodies to come together!

But in a region with a phase-space density of particle type A of $f_A(p,x)$ (number of particles per d^3x per d^3p), the number of $ABC \to DEF$ scatterings is:

$$
\begin{aligned}
N_{ABC\to DEF} = \int d^4x \int & \frac{d^3p_A d^3p_B d^3p_C}{(2\pi)^9 2E_A 2E_B 2E_C} f_A(x,p_A) f_B(x,p_B) f_C(x,p_C) \\
& \times \int \frac{d^3p_D d^3p_E d^3p_F}{(2\pi)^9 2E_D 2E_E 2E_F} (1 \pm f_D(x,p_D))(1 \pm f_E(x,p_E))(1 \pm f_F(x,p \\
& \times (2\pi)^4 \delta^4(p_A + p_B + p_C - p_D - p_E - p_F) |\mathcal{M}_{ABC\to DEF}|^2
\end{aligned}
$$

Here $(1 \pm f_{D,E,F})$ are Bose stimulation (+) or Pauli blocking ($-$) factors.

16: Summary

Unstable particles decay. If they live long compared to $\hbar/mc^2$ then there is a well-defined **decay rate** Γ.

The final state particles must match the energy and momentum of the initial particle. With this constraint, we must integrate over all possible final state momenta.

Two or more particles can scatter. They scatter more often if you have good aim: rate of scatterings is “cross-section” (physics) times “luminosity” (accelerator design)

The total cross-section involves an integral over all final state momenta, subject to energy-momentum conservation. “Physics” is incorporated in the **matrix element** $\mathcal{M}$.