Teilchenphysik:

Lecture 7: Feynman calculus

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Next step: decays and scattering in actual theories
But actual theories are too complicated.

We will set up technology with a *toy example*

- How do you write down the *contents* of a theory?
- How does that determine the way the constituents interact?
- What do real calculations of decays look like?
- What do real calculations of scattering look like?

2: The big picture so far

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Big questions of particle physics:

- What are the fundamental particles in nature?
  Answered in the first lecture. We'll come back...
- How do they interact with each other?
  Which ones are stable?
  How do the unstable ones decay?
  How do they scatter off of each other?

We have seen how to *help organize* these questions in terms of symmetries, and how to define and formulate decay and scattering.

Next: How do you really calculate any of this?

3: Why not to start with real examples

Real particle physics is complicated because the particles (almost) all carry nonvanishing spin. The ways the spins combine is complicated, with relativistic generalizations of Clebsch-Gordan coefficients.

Physicists learn by figuring out one piece at a time.
Let's first see how decays and scattering work for spin-0 particles,
and then worry about adding the complications of spin.

Therefore we will consider a **Toy Model**

4: Our toy model

Following the book, consider a theory of **three scalar fields** A, B, C:

$$m_A > m_B + m_C \qquad m_B > m_C$$

They interact via a three-field interaction.
All physics is determined by the *action*
in *field theories* the action is a spacetime integral of a **Lagrange density** $\mathcal{L}$:

$$S = \int dt\, L\,, \qquad L = \int d^3x\, \mathcal{L}\,, \quad \text{for us,} \quad \mathcal{L} = ... + gABC$$

with g some **coupling constant**

5: Feynman rules

Feynman developed a simple graphical method to keep track of all the ways particles can move and interact: the Feynman rules

Particles arrive initially

Particles interact at **vertices**

Particles propagate
from vertex to vertex

Particles fly out

6: Feynman rules and Momentum

How does energy-momentum move?

Particles bring in 4-momentum

It is conserved at **vertices**

It is carried along propagators

Particles carry out 4-momentum

7: Super simple example: A decay

The process $A \to BC$ is allowed. One leading diagram:

Feynman rules always give $-i$ times $(2\pi)^4\delta^4(p_{\text{in}} - p_{\text{out}})$ times **Matrix Element**

$$-ig(2\pi)^4\delta^4(p_A^\mu - p_B^\mu - p_C^\mu) = -i(2\pi)^4\delta^4(p_A^\mu - p_B^\mu - p_C^\mu)\mathcal{M} \qquad \mathcal{M} = g$$

8: A decay: calculation

Fermi's Golden Rule:

$$\Gamma_A = \frac{1}{2m_A} \int \frac{d^3p_B\, d^3p_C}{(2\pi)^6\, 2E_B\, 2E_C} (2\pi)^3 \delta^3(\vec{p}_B + \vec{p}_C)(2\pi)\delta(m_A - E_B - E_C) \times g^2$$

Here I separated the momentum and energy parts of the $\delta^4(..)$.
Start with baby case: $m_B = 0 = m_C$ (or $m_A \gg m_B, m_C$)

Use delta-function to perform $\int d^3\vec{p}_C$:

$$\vec{p}_C = -\vec{p}_B \qquad E_B = |\vec{p}_B| \qquad E_C = |\vec{p}_C| = E_B$$

Simplifies result:

$$\Gamma = \frac{g^2}{(2\pi)^2 2m_A} \int \frac{p_B^2 dp_B}{(2p_B)^2} \int d\Omega_B \times \delta(m_A - 2E_B)$$

9: Integral of a Delta Function

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$$\Gamma = \frac{g^2}{8\pi^2 m_A} \left(\int d\Omega_B \right) \left(\int \frac{dp_B}{4} \delta(m_A - 2E_B) \right)$$

CAREFUL!

$\int_0^\infty dx\, \delta(2-2x)$ Name $y = 2x$ and $\int_0^\infty \frac{2dx}{2}\, \delta(2-2x) = \int_0^\infty \frac{dy}{2}\, \delta(2-y) = \frac{1}{2}$

Don't forget constants, functional dependences inside delta functions.

In general: $$\int dx\, \delta(f(x)) = \sum_{x_i : f(x_i)=0} \frac{1}{|f'(x_i)|}$$

Meanwhile, $\int d\Omega_B = 4\pi$ as usual.

10: Decay rate: final result

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For the case $m_A \gg m_B, m_C$ we find:

$$\Gamma = \frac{g^2}{8\pi^2 m_A}\left(\int d\Omega_B\right)\left(\int \frac{dp_B}{4}\delta(m_A - 2p_B)\right)$$
$$= \frac{g^2}{8\pi^2 m_A} \times 4\pi \times \frac{1}{8} = \frac{g^2}{16\pi m_A}$$

The case where m_B, m_C are not so small is more complex. We need

$$\int \frac{p_B^2 dp_B}{2\sqrt{p_B^2 + m_B^2}\, 2\sqrt{p_B^2 + m_c^2}}\delta\left(m_A - \sqrt{p_B^2 + m_B^2} - \sqrt{p_B^2 + m_C^2}\right)$$

After some work (see book) we find:

$$|p_B| = \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}, \quad \Gamma = \frac{|p_B| g^2}{8\pi m_A^2}$$

11: Is the B stable?

Suppose $m_B > 2m_C$. There is enough energy for B to decay.
Can B decay into some C particles? Try drawing a diagram:

Why can't I do it? Discrete symmetry: $A \to -A$ and $B \to -B$.
Total number of A plus B particles stays either even or odd.

12: Higher order corrections?

Multi-particle decays

$\mathcal{O}(g^6)$

Loop effects

$\mathcal{O}(g^4)$

Consider first $BC \rightarrow BC$. *Two* diagrams!

What should we do? Add them.

14: Momenta

Let's write out all the momenta for this process.

Use $\delta^4(p_1+p_2-q)$ to set $q = p_1+p_2$ and use to rewrite $\delta^4(q-p_3-p_4)$ as $\delta^4(p_1+p_2-p_3-p_4)$

$$\mathcal{M} = i\frac{i(-ig)^2}{(p_1+p_2)^2 - m_A^2}$$

Use $\delta^4(p_1-p_4-q)$ to write $q = p_1-p_4$ and rewrite $\delta^4(p_2-p_3+q)$ as $\delta^4(p_1+p_2-p_3-p_4)$.

$$\mathcal{M} = i\frac{i(-ig)^2}{(p_1-p_4)^2 - m_A^2}$$

15: Mandelstam Variables

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Consider two incoming particles, (p_1, p_2) and (m_1, m_2) and two outgoing particles (p_3, p_4) and (m_3, m_4). Frequently encounter

$$c^2(p_1+p_2)^2 \equiv s = c^2(p_3+p_4)^2$$
$$c^2(p_1-p_3)^2 \equiv t = c^2(p_2-p_4)^2$$
$$c^2(p_1-p_4)^2 \equiv u = c^2(p_2-p_3)^2$$

Not all independent: $(p_1+p_2-p_3-p_4) = 0$ and $p_1^2 = c^2 m_1^2$, $p_2^2 = c^2 m_2^2$, $p_3^2 = c^2 m_3^2$, $p_4^2 = c^2 m_4^2$

and therefore (after some work) $\quad s + t + u = c^4(m_1^2 + m_2^2 + m_3^2 + m_4^2)$

For very relativistic systems $t < 0$ and $u < 0$, and

$$s = E_{\mathrm{cm}}^2 \text{ (always)}, \qquad t = -\frac{(1 - \cos\theta_{13})s}{2}, \qquad u = -\frac{(1 + \cos\theta_{13})s}{2}.$$

16: Processes and Mandelstam

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$BC \to BC$

$BB \to CC$

$$g^2 \left(\frac{1}{s - m_A^2} + \frac{1}{u - m_A^2} \right)$$

$$g^2 \left(\frac{1}{t - m_A^2} + \frac{1}{u - m_A^2} \right)$$

17: Symmetry factors

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Consider $BB \to CC$. *total* cross-section:

$$\sigma = \frac{1}{2E_1 2E_2 |v_1 - v_2|} \int \frac{d^3p_3\, d^3p_4}{(2\pi)^6\, 2E_1\, 2E_2} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)\, g^4 \left(\frac{1}{t - m_A^2} + \frac{1}{u - m_A^2} \right)^2$$

Q: What p_3, p_4 range should I integrate?

A: Only half! ($p_3 = x, p_4 = y$) equivalent to ($p_4 = x, p_3 = y$) and integrating over both is double-counting physically distinct final states. Either

- Make sure to integrate over half of the final momentum range, leaving out all "duplicate" integrations, or
- Integrate over everything but put in a "symmetry" factor $\frac{1}{2}$ to correct for double-counting.

If there were N final-state B particles, we would need a $1/N!$ factor or to integrate over only $1/N!$ of the "naive" phase space.

18: Doing the angular integral

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Let's compute total cross-section in $s \gg m_A^2$ limit in CM frame:

$$\sigma = \frac{1}{2E_1 2E_2 |v_1 - v_2|} \int \frac{d^3p_3\, d^3p_4}{(2\pi)^6\, 2E_1\, 2E_2} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)\, g^4 \left(\frac{1}{t - m_A^2} + \frac{1}{u - m_A^2} \right)^2$$

- $|v_1 - v_2| = 2$ and $2E_1 2E_2 = s$
- $t = (1 - \cos(\theta_{13}))s/2$ and $u = (1 + \cos(\theta_{13}))s/2$
- The $\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) = \delta(\vec{p}_3 + \vec{p}_4)$ forces $\vec{p}_4 = -\vec{p}_3$, $E_4 = E_3 = E_1 = E_{cm}/2$.
- The $|p_3|$ integral is just like in the 2-body decay.

$$\sigma = \frac{g^4}{2s} \frac{1}{4\pi^2} \int \frac{p_3^2\, dp_3}{4p_3^2} \delta(E_{cm} - 2p_3) 2\pi \int \sin(\theta)\, d\theta \left(\frac{2}{(1 - \cos\theta)s} + \frac{2}{(1 + \cos\theta)s} \right)^2$$

Integral actually diverges as $\theta \to 0$ or $\theta \to \pi$, but if we include m_A^2 it stays finite.

19: What about $BB \rightarrow BB$?

This process actually requires a “box” diagram:

Cross-section is $O(g^8)$ so for “small” g it is very suppressed!

20: Summary

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- Total rates or widths are phase space times matrix element
- **Feynman Rules** exist to turn cartoons into determinations of the matrix elements
- For a toy example, the decay Feynman rule is very simple
- Some processes are forbidden; others just arise at high order
- Two-body phase space is surprisingly simple
- Mandelstam variables are very useful in two-body scattering