Teilchenphysik:

Lecture 8: Spin 1/2, Dirac equation

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Time to deal with spin!

- Klein-Gordon equation – what does a spin-0 particle do?
- General expectations about spin in relativistic theory
- Need for matrix equations
- Solutions to Dirac: particles at rest
- Negative energy solutions? The Dirac sea

2: Where are we in the big picture

Last time we saw how to calculate things in a theory with scalar particles.

- From interactions to Feynman rules
- Drawing valid Feynman diagrams
- Using them to compute $\mathcal{M}$, Γ, $d\sigma/d\Omega$

But we didn't see

- How does the propagator really work? Where does it come from?
- What if the particles have spin?
  Where do the Clebsch-Gordan coefficients come into play?

Today: *quickly* address first point, start to *patiently* address the second.
Not the goal to derive. Just to motivate and explain.

3: Klein-Gordon equation

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Schrödinger equation:

$$E = \frac{p^2}{2m} + V \quad \text{and} \quad E = i\hbar\partial_t\,,\ p^i = -i\hbar\partial_i \quad \text{gives} \quad i\hbar\partial_t\psi = \frac{\hbar^2}{2m}(-i\partial_i)(-i\partial_i)\psi + V\psi$$

Note: $(c^{-1}\partial_t, \partial_i) = \partial_\mu$, eg, $p_\mu \to i\hbar\partial_\mu$ (recall, $\vec{p} = -p_i$)

Relativistically I expect

$$E^2 = p^2c^2 + m^2c^4 \qquad \text{so} \quad -\hbar^2\partial_0^2\psi = -\hbar^2\partial_i^2\psi + m^2c^2\psi$$

$$0 = \hbar^2 g^{\mu\nu}\partial_\mu\partial_\nu\psi + m^2c^2\psi$$

Klein-Gordon equation

4: Klein-Gordon equation and Lagrangian

All physics comes from the action, which comes from a Lagrange-density.
How does that work for Klein-Gordon?

$$S = \int d^4x \; \mathcal{L} = \int d^4x \;\; \frac{-1}{2} \psi \left(\hbar^2 g^{\mu\nu} \partial_\mu \partial_\nu + m^2 c^2 \right) \psi$$

Vary with respect to $\psi(y)$ and you get Klein-Gordon equation at point y.

We haven't explained where Feynman rules come from, but it's a fact that the thing in the denominator of the propagator rule is (minus) the coefficient between ψ factors. Momentum space: $\hbar\partial_\mu \rightarrow ip_\mu$....

Expression is rotationally invariant. Nöther's theorem $\rightarrow$ angular momentum; we find: $\vec{J} = \vec{x} \times \vec{p} = \vec{L}$ the orbital angular momentum. Spin-0.

5: How do I get spin-$\frac{1}{2}$ particles?

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First, are there spin-$\frac{1}{2}$ particles?
The story of spin was worked out nonrelativistically. Now we are relativistic.

Two stories

- What are possible *states?* (What particles are possible?)
- What are possible *fields?* (What are things making the particles, and how will Feynman rules work?)

I will give more detail than the book, because it's interesting.

6: Reminder: Rotations and Spin

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Think back to the problem of spin and rotations.

Symmetry: rotations. R_x, R_y, R_z which don't all commute.
Neither do the generators J_x, J_y, J_z.
But J^2 commutes with everything.

Look for simultaneous eigenvalues of J^2, J_z.

Ladder operators $2J_\pm = J_x \pm iJ_y$: J_z values are quantized.
Avoiding infinite eigenvalues: J^2 also quantized.

Derives rules that $J^2 = \hbar^2 j(j+1)$ etc.

7: Possible states: Weyl's analysis

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Full set of (spacetime) symmetries: 4-translations and Lorentz transforms. (rotations + boosts)
Generators P^μ, $J^{\mu\nu}$ don't all commute. But P^2 and W^2 do.[1] Look for P^2, W^2, W^z eigenvalues.

Name P^2 eigenvalue m^2. It's the state's mass (or CM mass for multiparticle states).

Go to the rest frame. Then $W^i = mJ^i$ with J^i the ordinary angular momentum in that frame. **Turns into precisely the usual nonrelativistic story.**

But massless particles are different. They have half-integer *helicity* but no rule says all helicity states need to exist!

[1] $W^\mu = \epsilon^{\mu\nu\alpha\beta} J_{\nu\alpha} P_\beta \simeq m\vec{J}$ is some thing we meet along the way...

8: Helicity

9: Helicity: summary

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Helicity is whether spin points along, or opposite, a particle's direction of motion.

You can "catch up with and pass" a massive particle.

Therefore both helicities must always be present.

For very high energy particles, the helicity is "almost" conserved.

For massless particles, it's logically possible to have just one helicity.

10: What about fields?

What is the relativistic, spin-$\frac{1}{2}$ version of Schrödinger function ψ and Schrödinger equation? Three ways to figure this out:

- **Dirac's way:** inspired, physically motivated, lucky guess.
- **Weyl's way:** exhaustive group-theoretical analysis
- **What I will do:** Tell you the answer, try to motivate why

It won't make sense to write one equation for each possible spin value. We see from helicity discussion that they can mix with each other. So we will need ψ to be a *column vector* and the equation to involve *matrices*

11: What Weyl found

Weyl did an exhaustive study of what fields are possible:

- They must transform under representations of Lorentz $SO(3,1)$
- Story similar to, but more complicated than, with $SU(2)$ spin
- The smallest non-scalar representations are 2-component (Weyl spinors)
- Massive particles: helicity discussion...spin-up and spin-down cannot be each other's antiparticles.
  - Option 1: Massive particles are their own antiparticles!
  - Option 2: need two Weyl doublets which are each others' antiparticles, for 4 states in total.
- For massless particles, the two helicities can be particle and antiparticle. But then the "particle" state has only one possible helicity. (neutrinos)

Massive particles which are not their own antiparticles require two 2-component spinors – four states in total. This is solution found by Dirac

12: The Dirac equation

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Column vector ψ. Matrices γ^μ. Dirac equation

$$\left(\gamma^\mu p_\mu - mc\right)\psi = 0 \qquad \text{with} \quad p_\mu = i\hbar\partial_\mu$$

What should γ^μ be? Multiply both sides by $\gamma^\nu p_\nu + mc$

$$(\gamma^\nu p_\nu + mc)\left(\gamma^\mu p_\mu - mc\right)\psi = 0 = \left(p_\mu p_\nu \gamma^\mu \gamma^\nu - m^2c^2\right)\psi$$

I hope this turns into $(g^{\mu\nu} p_\mu p_\nu - m^2c^2)\psi = 0$ so $E^2 = \vec{p}^2c^2 + m^2c^4$.
Note: $p_\mu p_\nu$ is symmetric in $\mu \leftrightarrow \nu$. I can *symmetrize* the $\gamma^\mu\gamma^\nu$.
Therefore, it works iff the γ^μ obey the *Clifford Algebra*

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}$$

In $2N$ dimensions, this requires matrices at least 2^N in size.
so, in 4 dimensions, matrices are $2^2 = 4$ on a side, eg, 4×4

13: What are Gamma Matrices?

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Here are some matrices which work!

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \quad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Break into 2×2 blocks:

$$\gamma^0 = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} \qquad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$

with σ^i the Pauli matrices and **1** the 2×2 identity matrix.
Also each 0 is a 2×2 block of 0's.

Are these the only matrices which work? NO!
If S is invertible and γ^μ work, then $S^{-1}\gamma^\mu S$ also work. But let's use these.

14: What do solutions mean?

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Suppose I find a solution to Dirac equation: column vector

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad \text{with} \quad \left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi = 0$$

What does it represent? Careful: ψ is **not** a 4-vector!

A possible wave function for a spin-$\frac{1}{2}$ particle, mass m.

Just for fun: add a potential

$$i\hbar\gamma^\mu\partial_\mu\psi = (mc + V/c)\psi\,, \qquad V(r) = -\frac{\alpha}{r}$$

Coulomb potential for electron about a proton.

Solutions found by Dirac, get the hydrogen fine structure right.

15: Dirac particle at rest

Let's be very modest and look for a free, solution $\psi(x)$ which does not vary in space, $\partial_i \psi(x) = 0$:

$$i\hbar\gamma^0 \partial_t \psi = mc^2 \psi$$

Four possible solutions:

$$\psi = e^{-i\hbar mc^2 t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \psi = e^{-i\hbar mc^2 t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \psi = e^{+i\hbar mc^2 t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \psi = e^{+i\hbar mc^2 t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

First two: spin-up and spin-down components of spin doublet with $E = mc^2$.
Second two: spin-down and spin-up components of doublet with $E = -mc^2$.

WHAAAAATTT???

16: Dirac Sea

Dirac had an elegant explanation. Consider E as function of $\vec{p}$:

$E^2 = p^2c^2 + m^2c^4$ has positive, negative branch.
Two solutions (spin states) on each branch.

Spin-$\frac{1}{2}$ are fermions: zero or one particle in each possible state.
What if the negative-energy states are all filled? **Dirac Sea**

17: Summary

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Relation between energy, momentum and time, space derivatives:

$$(E/c, \vec{p}) = p^\mu \to g^{\mu\nu} i\hbar\partial_\mu \,, \quad \partial_\mu = (c^{-1}\partial_t, \vec{\nabla})$$

Naively insert into $E^2 = p^2c^2 + m^2c^4$: Klein-Gordon equation
$\hbar^2 g^{\mu\nu}\partial_\mu\partial_\nu\psi = -m^2c^2\psi$

Relativistically, massive particles fill out spin multiplets. Massless particles have fixed helicity; not all helicities need exist.

Massive spin-$\frac{1}{2}$ fields represented by 4-component Dirac vector ψ.
Obeys **Dirac equation** $i\hbar\gamma^\mu\partial_\mu\psi = mc\psi$
Gamma matrices γ^μ are 4×4 and obey Clifford algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}$
Solutions have $E = \pm\sqrt{m^2c^4 + p^2c^2}$. Negative energy: filled states (**Dirac Sea**)