there is an external electric or magnetic field, say in the z-direction. The rotational symmetry is now manifestly broken; as a result, the $(2j+1)$-fold degeneracy is no longer expected and states characterized by different m-values no longer have the same energy. We will examine how this splitting arises in Chapter 5.

SO(4) Symmetry in the Coulomb Potential

A fine example of continuous symmetry in quantum mechanics is afforded by the hydrogen atom problem and the solution for the Coulomb potential. We carried out the solution to this problem in Section 3.7, where we discovered that the energy eigenvalues in (3.7.53) show the striking degeneracy summarized in (3.7.56). It would be even more striking if this degeneracy were just an accident, but indeed, it is the result of an additional symmetry that is particular to the problem of bound states of $1/r$ potentials.

The classical problem of orbits in such potentials, the Kepler problem, was of course well studied long before quantum mechanics. The fact that the solution leads to elliptical orbits that are *closed* means that there should be some (vector) constant of the motion that maintains the orientation of the major axis of the ellipse. We know that even a small deviation from a $1/r$ potential leads to precession of this axis, so we expect that the constant of the motion we seek is in fact particular to $1/r$ potentials.

Classically, this new constant of the motion is

$$\mathbf{M} = \frac{\mathbf{p}\times\mathbf{L}}{m} - \frac{Ze^2}{r}\mathbf{r} \tag{4.1.19}$$

where we refer to the notation used in Section 3.7. This quantity is generally known as the *Lenz vector* or at times as the *Runge-Lenz vector*. Rather than belabor the classical treatment here, we will move on to the quantum-mechanical treatment in terms of the symmetry responsible for this constant of the motion.

This new symmetry, which is called SO(4), is completely analogous to the symmetry SO(3) studied in Section 3.3. That is, SO(4) is the group of rotation operators in *four* spatial dimensions. Equivalently, it is the group of orthogonal 4×4 matrices with unit determinant. Let us build up the properties of the symmetry that leads to the Lenz vector as a constant of the motion, and then we will see that these properties are those we expect from SO(4).

Our approach closely follows that given by Schiff (1968), pp. 235–39. We first need to modify (4.1.19) to construct a Hermitian operator. For two Hermitian vector operators $\mathbf{A}$ and $\mathbf{B}$, it is easy to show that $(\mathbf{A}\times\mathbf{B})^\dagger = -\mathbf{B}\times\mathbf{A}$. Therefore, a Hermitian version of the Lenz vector is

$$\mathbf{M} = \frac{1}{2m}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) - \frac{Ze^2}{r}\mathbf{r}. \tag{4.1.20}$$

It can be shown that $\mathbf{M}$ commutes with the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{Ze^2}{r}; \tag{4.1.21}$$

that is,

$$[\mathbf{M}, H] = 0, \tag{4.1.22}$$

so indeed $\mathbf{M}$ is a (quantum-mechanical) constant of the motion. Other useful relations can be proved, namely

$$\mathbf{L}\cdot\mathbf{M} = 0 = \mathbf{M}\cdot\mathbf{L} \tag{4.1.23}$$

and $$\mathbf{M}^2 = \frac{2}{m} H \left(\mathbf{L}^2 + \hbar^2\right) + Z^2 e^4. \tag{4.1.24}$$

In order to identify the symmetry responsible for this constant of the motion, it is instructive to review the algebra of the generators of this symmetry. We already know part of this algebra:

$$\left[L_i, L_j\right] = i\hbar\varepsilon_{ijk} L_k, \tag{4.1.25}$$

which we wrote earlier as (3.6.2) in a notation where repeated indices (k in this case) are automatically summed over components. One can also show that

$$\left[M_i, L_j\right] = i\hbar\varepsilon_{ijk} M_k, \tag{4.1.26}$$

which in fact establish $\mathbf{M}$ as a vector operator in the sense of (3.11.8). Finally, it is possible to derive

$$\left[M_i, M_j\right] = -i\hbar\varepsilon_{ijk} \frac{2}{m} H L_k. \tag{4.1.27}$$

To be sure, (4.1.25), (4.1.26), and (4.1.27) do not form a closed algebra, due to the presence of H in (4.1.27), and that makes it difficult to identify these operators as generators of a continuous symmetry. However, we can consider the problem of specific bound states. In this case, the vector space is truncated only to those that are eigenstates of H, with eigenvalue $E < 0$. In that case, we replace H with E in (4.1.27), and the algebra is closed. It is instructive to replace $\mathbf{M}$ with the scaled vector operator

$$\mathbf{N} \equiv \left(-\frac{m}{2E}\right)^{1/2} \mathbf{M}. \tag{4.1.28}$$

In this case we have the closed algebra

$$\left[L_i, L_j\right] = i\hbar\varepsilon_{ijk} L_k, \tag{4.1.29a}$$

$$\left[N_i, L_j\right] = i\hbar\varepsilon_{ijk} N_k, \tag{4.1.29b}$$

$$\left[N_i, N_j\right] = i\hbar\varepsilon_{ijk} L_k. \tag{4.1.29c}$$

So what is the symmetry operation generated by the operators $\mathbf{L}$ and $\mathbf{N}$ in (4.1.29)? Although it is far from obvious, the answer is "rotation in *four* spatial dimensions." The first clue is in the number of generators, namely six, each of that should correspond to rotation about some axis. Think of a rotation as an operation that mixes two orthogonal axes. Then, the number of generators for rotations in n spatial dimensions should be the number of combinations of n things taken two at a time, namely $n(n-1)/2$. Consequently, rotations in two dimensions require one generator—that is, L_z. Rotations in three dimensions require three generators, namely $\mathbf{L}$, and four-dimensional rotations require six generators.

It is harder to see that (4.1.29) is the appropriate algebra for this kind of rotation, but we proceed as follows. In three spatial dimensions, the orbital angular-momentum operator (3.6.1) generates rotations. We saw this clearly in (3.6.6), where an infinitesimal z-axis rotation on a state $|\alpha\rangle$ is represented in a rotated version of the $|x,y,z\rangle$ basis. This was just a consequence of the momentum operator being the generator of translations in space. In fact, a combination like $L_z = xp_y - yp_x$ indeed mixes the x-axis and y-axis, just as one would expect from the generator of rotations about the z-axis.

To generalize this to four spatial dimensions, we first associate (x,y,z) and (p_x,p_y,p_z) with (x_1,x_2,x_3) and (p_1,p_2,p_3). We are led to rewrite the generators as $L_3 = \tilde{L}_{12} = x_1p_2 - x_2p_1$, $L_1 = \tilde{L}_{23}$, and $L_2 = \tilde{L}_{31}$. If we then invent a new spatial dimension x_4 and its conjugate momentum p_4 (with the usual commutation relations), we can define

$$\tilde{L}_{14} = x_1p_4 - x_4p_1 \equiv N_1, \tag{4.1.30a}$$

$$\tilde{L}_{24} = x_2p_4 - x_4p_2 \equiv N_2, \tag{4.1.30b}$$

$$\tilde{L}_{34} = x_3p_4 - x_4p_3 \equiv N_3. \tag{4.1.30c}$$

It is easy to show that these operators N_i obey the algebra (4.1.29). For example,

$$\begin{aligned}[N_1, L_2] &= [x_1p_4 - x_4p_1, x_3p_1 - x_1p_3] \\ &= p_4[x_1,p_1]x_3 + x_4[p_1,x_1]p_3 \\ &= i\hbar(x_3p_4 - x_4p_3) = i\hbar N_3.\end{aligned} \tag{4.1.31}$$

In other words, this is the algebra of four spatial dimensions. We will return to this notion in a moment, but for now we will press on with the degeneracies in the Coulomb potential that are implied by (4.1.14).

Defining the operators

$$\mathbf{I} \equiv (\mathbf{L}+\mathbf{N})/2, \tag{4.1.32}$$

$$\mathbf{K} \equiv (\mathbf{L}-\mathbf{N})/2, \tag{4.1.33}$$

we easily can prove the following algebra:

$$\left[I_i, I_j\right] = i\hbar\varepsilon_{ijk}I_k, \tag{4.1.34a}$$

$$\left[K_i, K_j\right] = i\hbar\varepsilon_{ijk}K_k, \tag{4.1.34b}$$

$$\left[I_i, K_j\right] = 0. \tag{4.1.34c}$$

Therefore, these operators obey independent angular-momentum algebras. It is also evident that $[\mathbf{I}, H] = [\mathbf{K}, H] = 0$. Thus, these "angular momenta" are conserved quantities, and we denote the eigenvalues of the operators $\mathbf{I}^2$ and $\mathbf{K}^2$ by $i(i+1)\hbar^2$ and $k(k+1)\hbar^2$, respectively, with $i, k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

Because $\mathbf{I}^2 - \mathbf{K}^2 = \mathbf{L}\cdot\mathbf{N} = 0$ by (4.1.23) and (4.1.28), we must have $i = k$. On the other hand, the operator

$$\mathbf{I}^2 + \mathbf{K}^2 = \frac{1}{2}\left(\mathbf{L}^2 + \mathbf{N}^2\right) = \frac{1}{2}\left(\mathbf{L}^2 - \frac{m}{2E}\mathbf{M}^2\right) \tag{4.1.35}$$

leads, with (4.1.24), to the numerical relation

$$2k(k+1)\hbar^2 = \frac{1}{2}\left(-\hbar^2 - \frac{m}{2E}Z^2e^4\right). \tag{4.1.36}$$

Solving for E, we find

$$E = -\frac{mZ^2e^4}{2\hbar^2}\frac{1}{(2k+1)^2}. \tag{4.1.37}$$

This is the same as (3.7.53) with the principal quantum number n replaced by $2k+1$. We now see that the degeneracy in the Coulomb problem arises from the two "rotational" symmetries represented by the operators $\mathbf{I}$ and $\mathbf{K}$. The degree of degeneracy, in fact, is $(2i+1)(2k+1) = (2k+1)^2 = n^2$. This is exactly what we arrived at in (3.7.56), except it is now clear that the degeneracy is no accident.

It is worth noting that we have just solved for the eigenvalues of the hydrogen atom without ever resorting to solving the Schrödinger equation. Instead, we exploited the inherent symmetries to arrive at the same answer. This solution was apparently first carried out by Pauli.

In the language of the theory of continuous groups, which we started to develop in Section 3.3, we see that the algebra (4.1.29) corresponds to the group SO(4). Furthermore, rewriting this algebra as (4.1.34) shows that this can also be thought of as two independent groups SU(2)—that is, SU(2) × SU(2). Although it is not the purpose of this book to include an introduction to group theory, we will carry this a little further to show how one formally carries out rotations in n spatial dimensions—that is, the group SO(n).

Generalizing the discussion in Section 3.3, consider the group of $n \times n$ orthogonal matrices R that carry out rotations in n dimensions. They can be parameterized as

$$R = \exp\left(i\sum_{q=1}^{n(n-1)/2}\phi^q\tau^q\right), \tag{4.1.38}$$

where the τ^q are purely imaginary, antisymmetrical $n \times n$ matrices—that is, $(\tau^q)^T = -\tau^q$—and the ϕ^q are generalized rotation angles. The antisymmetry condition ensures that R is orthogonal. The overall factor of i implies that the imaginary matrices τ^q are also Hermitian.

The τ^q are obviously related to the generators of the rotation operator. In fact, it is their commutation relations that should be parroted by the commutation relations of these generators. Following along as in Section 3.1, we compare the