

Lecture 16: Intervals, Scales, and Tunings

Rather than using any available frequency, most musical traditions use several fixed frequencies. These should be chosen (or have been chosen) to ensure the presence of lots of harmonies, that is, pairs of notes with overlapping harmonics. In particular, it is *essential* that if a frequency f is a standard note, that twice the frequency, $2 \times f$, also be a standard note (one **octave** up). It is also very nice that, if f is a standard note, the note at 1.5 times that frequency (up by one **fifth**), also be a standard note. Recall that our frequency sense cares about the logarithm of frequency.

There is an amazing numerical coincidence:

$$2^{7/12} \simeq 1.5 = \frac{3}{2} \quad (\text{actually it is } 1.4983\dots)$$

which means that if we divide the octave into 12 evenly spaced (in logarithm) frequency steps, then notes separated by 7 of those steps will be related by a musical fifth, the ratio 1.5. Further, because

$$\frac{2}{3/2} = \frac{4}{3}, \quad \text{so} \quad 2^{5/12} \simeq \frac{4}{3} \quad (\text{actually it's } 1.3348\dots)$$

we will also find that notes separated by 5 of these steps are spaced by a ratio of 4/3, the musical **fourth**. (We will see where these names come from in a moment). Furthermore, the following approximation, while not quite so accurate, is not too bad:

$$2^{4/12} \simeq 1.25 = \frac{5}{4} \quad (\text{really } 1.2599) \quad \text{and} \quad 2^{3/12} \simeq 1.20 = \frac{6}{5} \quad (\text{really } 1.1892)$$

Therefore such spacing of notes also gives the two next most important intervals, called the **major third** and **minor third**, fairly well, though not perfectly.

Because of these numerical accidents, music designed around 12 (approximately) even divisions of the octave, that is, frequencies which are some standard times 1, $2^{1/12}$, $2^{2/12}$, $2^{3/12}$, \dots , $2^{12/12} = 2$, is a good idea, and several musical traditions have converged on this idea: the Western musical tradition, the Eastern pentatonic tradition, and West African tradition (which uses $2^{1/24}$, half the spacing). In Western tradition, the spacing between notes separated in frequency by a factor of $2^{1/12}$ is called a **half-step** and we will use this nomenclature from now on:

Two notes of frequency f_1 and f_2 are said to be separated by a **half-step**, with f_2 a half-step higher than f_1 , if $f_2/f_1 = 2^{1/12}$, or $\log_2(f_2/f_1) = 1/12$ (which is the same).

How should these notes be used in music? It turns out that using all of them encounters numerous dissonances, as well as the consonances we designed for. In particular, these intervals are quite dissonant:

- notes separated by one half-step, $f_2/f_1 = 2^{1/12} = 1.05946$. The notes are so close that their critical bands badly overlap. Similarly, 13 half-steps is bad because of the 2'nd harmonic of the lower note.
- notes separated by 11 half-steps, $f_2/f_1 = 2^{11/12} = 1.8877$. The lower note's second harmonic partly overlaps the upper note, causing dissonance.
- notes separated by 6 half-steps, $f_2/f_1 = 2^{6/12} = 1.4142$. Several higher overtones come too close to overlapping.

Therefore it is a good idea to only use some of the 12 available notes, to make these overlaps rare. (Early to mid 20'th century composers called 12-tonalists experimented with using all 12 without prejudice. While the resulting music had some success within the experimental music community, it never gained wide popularity; the public does not want that much dissonance.)

In Western tradition, this problem is solved by using the following notes: some chosen fixed note, call it **Do**, and the notes separated from it by 2, 4, 5, 7, 9, 11, and 12 (and then 12+2, 12+4, 12+5, 12+7, ...) half-steps—or, if you prefer, moving forward by 2 half steps, 2 half steps, 1 halfstep, 2 half-steps, 2 half-steps, 2 half-steps, 1 half-step:

frequency	# half-steps	name
$f_0 \times 2$	12	Do
$f_0 \times 2^{11/12} = 1.8877$	11	Ti
$f_0 \times 2^{9/12} = 1.6818$	9	La
$f_0 \times 2^{7/12} = 1.4983$	7	Sol
$f_0 \times 2^{5/12} = 1.3348$	5	Fa
$f_0 \times 2^{4/12} = 1.2599$	4	Mi
$f_0 \times 2^{2/12} = 1.1225$	2	Re
$f_0 \times 1$	0	Do

By making two of the spacings one half step and the rest of them two, there are lots of pairs separated by 5 or 7 half-steps, and only one combination separated by 6 half-steps. To clarify, at the top, the note one octave above the starting note is also called Do, and one then repeats the spacings of intervals and the names of the notes. That is,

# half-steps	name
...	...
26	Fa
24	Do
23	Ti
21	La
19	Sol
17	Fa
16	Mi
14	Re
12	Do
11	Ti
9	La
7	Sol
5	Fa
4	Mi
2	Re
0	Do
-1	Ti
...	...

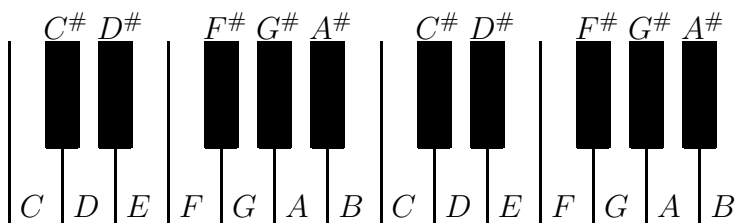
Incidentally, the fastest way to describe pentatonic is to say it is like the above, but never using Mi or Ti. That is, instead of the intervals going 2 2 1 2 2 2 1, they go 2 3 2 2 3. This avoids *ever* encountering 1, 6, or 11 half-steps, the three most dissonant intervals. The result is music which is highly consonant, but considered by Western ears to be a little boring (all resolution, no tension).

I have not said anything about what frequency Do should be. We can choose it to be whatever we want—say, 200 Hertz, or 210 Hertz, or 317.5 Hertz, whatever. However, because instruments like pianos are difficult to tune, it is a good idea to choose one special standard value. Then tune the piano, and allow someone to start with “Do” being either the standard value or anything separated by some number of half-steps from that standard value.

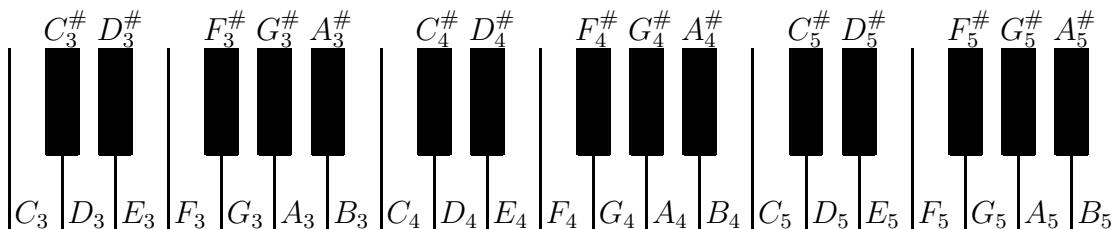
The standard value for Do is named C , and Re etc are named Re= D , Mi= E , Fa= F , Sol= G , La= A , and Ti= B . It has become conventional to tune A to 440 Hertz. In that case we name the notes as follows:

halfsteps above Do	Name
0	C
1	$C^\# = D^\flat$
2	D
3	$D^\# = E^\flat$
4	E
5	F
6	$F^\# = G^\flat$
7	G
8	$G^\# = A^\flat$
9	A
10	$A^\# = B^\flat$
11	B
12	C
...	...

On the piano keyboard, it looks like this:

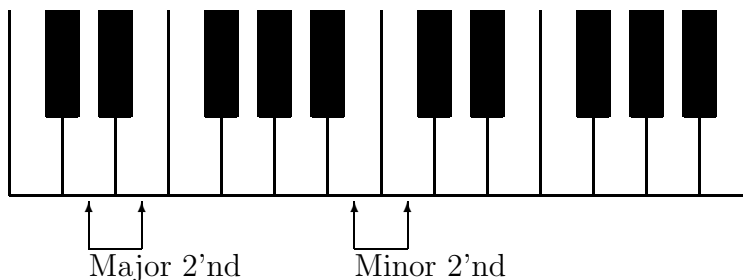


Now the problem is that there are actually several notes named A : the note at 440 Hertz, and all notes separated by it by some number of octaves: going down, 220 Hertz, 110 Hertz, 55 Hertz, 27.5 Hertz, and going up, 880 Hertz, 1760 Hertz, 3520 Hertz, and so forth. We distinguish these by putting little subscripts on them, as follows. The lowest C which appears on a piano keyboard is called C_1 ; the notes going up from it are D_1 , E_1 , and so forth, until B_1 ; then the next C is called C_2 . That is, as you go up the keyboard, each time you reach C , you add one to the little index. Middle C is C_4 , at about 261 Hertz. A_4 is the one with a frequency of 440 Hertz. **Careful:** the index changes at C , *not* at A . That is too bad, because the lowest note on a piano keyboard is actually an A (at 27.5 Hertz), which is unfortunately called A_0 .



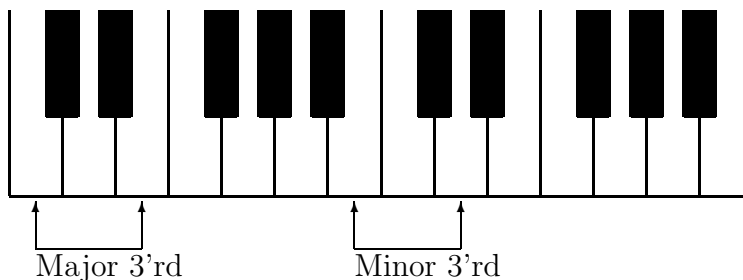
The nomenclature for intervals is as follows: the interval (frequency ratio) between two white keys on the keyboard is the number of white keys, starting with the lower key, counting all the ones between them, and *counting* the top key. That is, if two keys are next to each other, the interval is a **second**; if they are separated by 1 key, it is a **third**; by two keys, a **fourth**; and so forth. Because there are two places where there is no black key between white keys, most intervals can occur in one of two ways; with a larger number of black keys between, or with a smaller number between. The way with more black keys between, which means that there are more half-steps, is called a **major** interval; with fewer, a **minor** interval. Since this is confusing, I will illustrate the first few examples.

Look first at the seconds (neighboring keys):



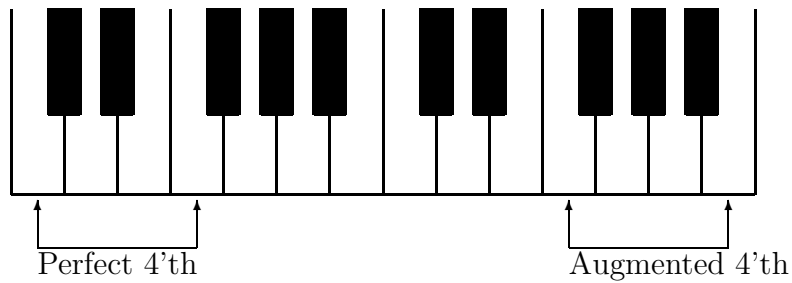
There are 7 ways of taking pairs of neighboring keys. In 5 of them, there is a black key in between, so they are separated by 2 half-steps. In two cases there is no black key, so they are separated by only 1 half-step. Therefore a separation of 2 half-steps is called a **major second** (also a **whole step**); a separation of one half-step is a **minor second** (also **half-step**).

For the thirds:



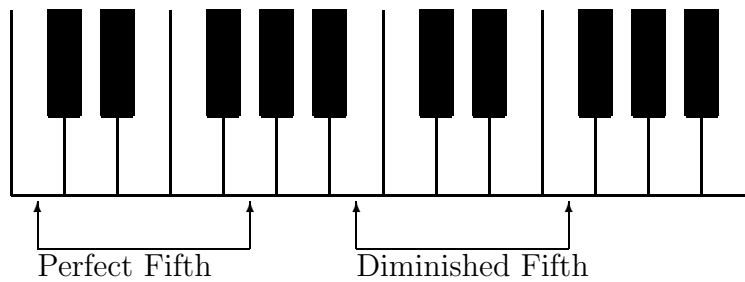
4 cases involve 3 half-steps, and 3 cases involve 4 half-steps. Therefore a separation of 3 half-steps is called a **minor third**, and 4 half-steps is called a **major third**.

For fourths:



All but one case involves 5 half-steps, which is also an ideal musical interval. Since it is so common and so nice, it is called a **perfect 4'th**. The other case is 6 half-steps, called an **augmented 4'th**.

For the fifths:



All but one case involves 7 half-steps, again a special interval called the **perfect fifth**. The one case with 6 half-steps is called a **diminished fifth**. This is the *same thing* as an augmented fourth. Just for fun, it has an extra name, the **tritone** (I don't know why).

For the sixths, they are either 8 or 9 half-steps; 8 is a **minor sixth**, 9 a **major sixth**. For the sevenths, it is either 10 or 11; a **minor 7'th** or a **major 7'th**. For the eighth, it is *always* 12 steps, called a **perfect 8'th**, an **8'th**, or most commonly an **octave**. One can define 9'ths, 10'ths, and so forth, but they are just 12 half-steps plus the 2'nds, 3'rds, and so forth.

In short:

# semitones	Name
1	minor second (2'nd)
2	major second (2'nd)
3	minor third (3'rd)
4	major third (3'rd)
5	perfect fourth (4'th)
6	augmented 4'th, diminished 5'th, tritone
7	perfect fifth (5'th)
8	minor sixth (6'th)
9	major sixth (6'th)
10	minor seventh (7'th)
11	major seventh (7'th)
12	octave

Tunings

The way I have described tuning—using $2^{1/12}$ for the half-steps—is called **equal tempered tuning**. It is pretty standard, because keyboard instruments cannot be re-tuned at will, and it is common to play in many different keys. However, it has some disadvantages; none of the perfect 4'th and 5'ths are actually frequency multiples of $4/3$ and $3/2$, and none of the major and minor thirds are multiples of $5/4$ or $6/5$. Historically (and in practice!) there are a few other ways of tuning, where one does *not* make all the intervals the same size, in order to make these intervals more perfect.

Pythagorean tuning

One makes all octaves exact factors of 2, and all but 1 of the 5'ths exact factors of $3/2$. This is the oldest and most traditional way of tuning, and it has the advantage that it is easy to do; one uses beats to determine if two notes are in tune. If there are beats, they are not in tune, and must be changed until they are in tune.

The tuning works like this: one first tunes A_4 to be 440 Hertz (say). Then tune the notes up by a fifth, E_5 , by making it $3/2$ the frequency of the A . Tune the note up a fifth from E_5 , B_5 , to be $3/2$ the frequency of E_5 . Also, you can tune downwards from A_4 ; down a fifth is D_4 , tuned to have $2/3$ the frequency of A_4 ; down from it is G_3 , tuned to have $2/3$ the frequency of D_4 , and so on. The notes differing by an octave are tuned by using that an octave should be an exact factor of 2.

The series of notes you get, by going up and down by fifths, is called the “circle of 5'ths”,

because after going 12 fifths you come back to the same letter note you started with:

$$\dots G^\# = A^b \quad E^b \quad B^b \quad F \quad C \quad G \quad D \quad A \quad E \quad B \quad F^\# \quad C^\# \quad G^\# = A^b \dots$$

This way of tuning was discovered by Pythagoras (yes, the triangle dude) about 2700 years ago. It would work perfectly, except that the $G^\#$ you get to at the top does not differ from the A^b you get to at the bottom by the 7 octaves that it should. That would happen if

$$\left(\frac{3}{2}\right)^{12} \stackrel{?}{=} 2^7 \quad \text{or} \quad 3^{12} \stackrel{?}{=} 2^{19}$$

which it does not quite: they are actually different by about 1.36%. That means that one of the 12 fifths has to be mis-tuned. It is usually chosen to be a fifth between two of the sharped or flatted notes, on the theory that it will not be played very often.

Most of the time one plays in keys where the “wrong” 5’t h is not used, and everything is fine. However, when you do play in a key where that 5’t h is used, it sounds wrong. There are sound file examples as extras for this lecture.

Just tuning

This is a more ambitious attempt to get the thirds, as well as the fifths, to be in tune. If you tune an instrument this way and play in the “right” key, it sounds great. Play in the “wrong” key, and it sounds painfully bad.

The problem is that the “circle of thirds” runs into trouble much faster than the circle of fifths. Going up by major thirds from C , one encounters $C, E, G^\#, C$. If we force each interval to be a factor of $5/4$, then the difference from C to C would be $(5/4)^3 = 125/64$, which is *not* $2 = 128/64$. In fact, it is pretty far off. That means that one major third out of every three has to be out of tune, by a big margin. No matter how you decide which ones to get “right,” there are many keys which will require using the ones which are wrong. If you listen to the music accompanying this lecture, you will see that there are several keys where the just tuned music sounds beastly.

For these reasons, usual practice is,

- Tune instruments in equal temperament (all half-steps equal, $2^{1/12}$);
- Cheat whenever your ear tells you to, particularly by playing the higher tone in a major third flat and the higher tone in a minor third sharp.