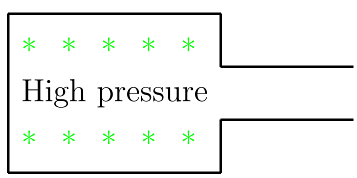
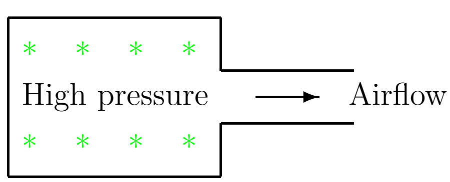
Lecture 19: Resonance and Cavities

A key idea–perhaps the key idea–in musical instruments is the idea of resonance. Resonance is what you call it when some system can store energy two different ways, and energy goes back and forth from one form to the other, generally in a sine wave pattern. The example you are probably most familiar with is a mass suspended from a spring. You might be even more familiar, from your childhood, with someone swinging on a swing. In this case, the two ways energy can be stored are, motion of the person on the swing, and the person being raised up into the air (energy stored in gravitational potential). Starting when the person is at the top of the swinging motion, gravity pulls them down and therefore forward. As they gather speed, the energy is moving from gravitational potential into their motion. At the bottom of the swinging motion, all the energy is in motion. But once in motion, you remain so until something stops you, so you keep right on going forward and upward. The energy is then getting taken out of motion into height (gravitational potential) again. At the top of the swing pattern, the whole thing repeats but with backward motion. If it were not for air resistance, losses in the swing chains, and so on, you would swing back and forth forever.

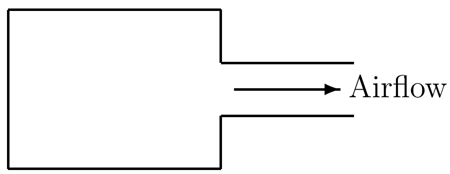
The same idea of resonance comes up in virtually every musical instrument. Since we have been studying sound, air pressure, and air motion, let us look at a nice example involving these ideas. Consider a wine bottle, or any other bottle with a big wide part and a long thin neck. [In physics we call something like this a Helmholtz resonator, after the physicist who first studied one carefully.] Suppose for some reason that the air inside the bottle is initially compressed. What happens to the air in the neck of the bottle?



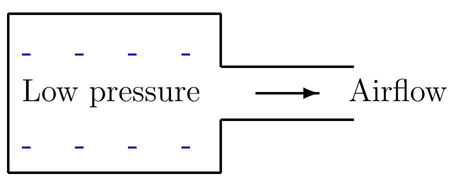
The air in the neck is pushed down by the pressure of the air outside. It is pushed up by the pressure of the air inside. Since that air is compressed, it pushes harder than the air outside. Therefore the air in the neck is pushed upwards. It accelerates upwards. If initially at rest, it will start to move, gathering speed with time. Therefore, an instant later, the situation will be like this:



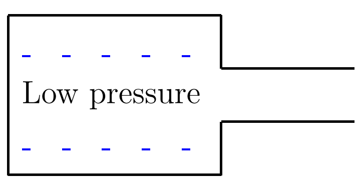
When the air in the neck is rising like this, there is a net flow of air, from the body of the bottle into the air outside the bottle. As air flows out of the body of the the bottle, the pressure inside will drop. After a moment, the pressure inside the bottle has fallen to be the same as the air pressure outside, atmospheric pressure:



However, once in motion, the air in the neck will remain in motion. Nothing needs to push it. Something has to push it the other way to get it to stop. This means that the air in the neck will keep flowing upwards, emptying out air from the bottle and bringing its pressure *below* atmospheric:

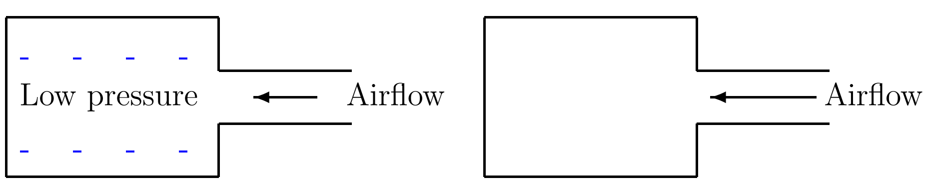


At this point, the air outside the bottle is pushing the neck air downwards more strongly than the air in the bottle is pushing it up. The air in the neck therefore slows down. After another moment, it has come to a stop:

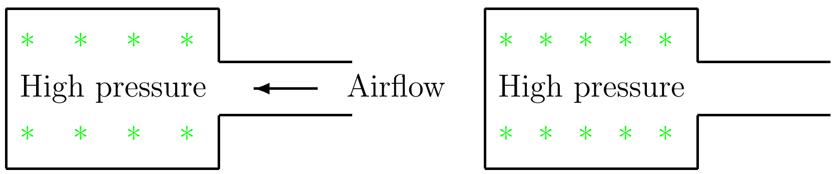


At this point the situation is exactly the opposite of what it was at the beginning.

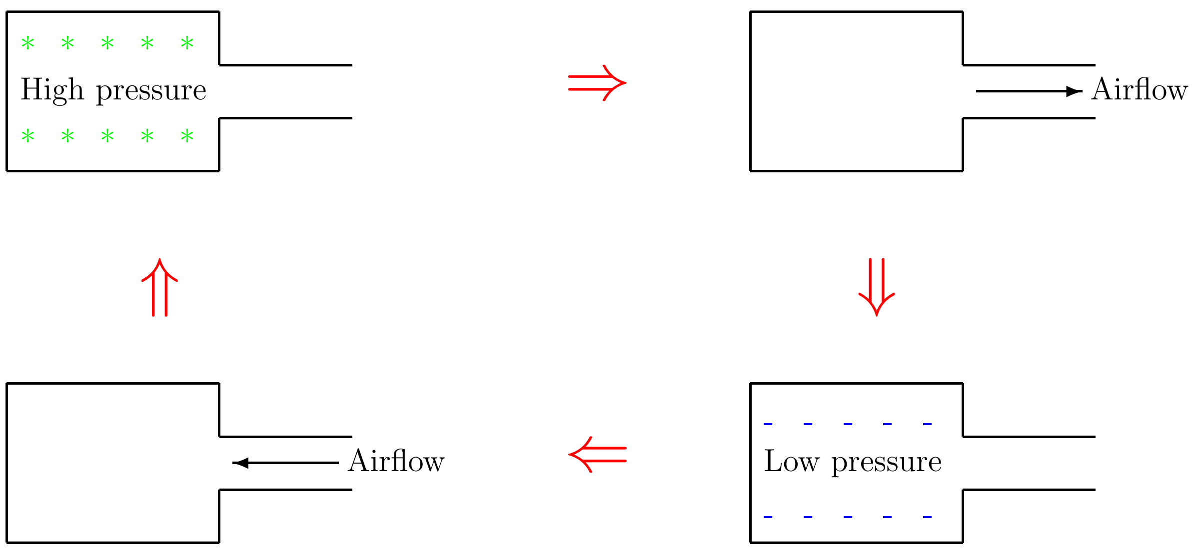
Now the air in the neck will continue to be pushed downwards, and will start flowing into the bottle. After two moments, it will have put the pressure in the bottle back up to normal:



Again the air keeps moving into the bottle until something actually stops it. Therefore it will re-pressurize the air inside the bottle. As the air pressure inside the bottle becomes more than outside, the air in the neck is again being pushed upwards, and will slow down and stop:



At this point, the situation is exactly as it was at the beginning. Therefore, the whole process will now repeat. This produces *periodic* changes in the pressure and air motion. Outside the bottle, the alternate outward and inward flows of air from the bottle will become a sound wave, with a period equal to how long it takes the process to repeat. In other words, the bottle goes round and round through the 4 stages of the resonance:



Let us try to figure out how long it takes for this resonance to go through one full cycle. This is going to depend on three parameters describing the bottle:

- V, the volume of the bottle;
- A, the cross-sectional area of the tube;
- l, the length of the tube.

Further, I will define ΔP to be the peak overpressure in the bottle, and v to be the peak air velocity in the neck. The period of the resonance won't depend on these, but they are useful in writing down the equations. Also, as usual I have to know the density of air, $\rho_{\rm air}$, and atmospheric pressure, $P_{\rm atmos}$.

I will *not* do a true derivation, but I will just make an estimate. A real derivation would not be much harder *if* I were willing to use calculus. Let us start by estimating how long it takes, to go from the bottle being pressurized without air flow, to it being all airflow. We begin by seeing how fast the airflow picks up. The force on the air in the neck is,

$$F = P_{\rm bottle} A - P_{\rm outside} A = \Delta P \; A$$

which causes an acceleration:

$$F = ma \quad \rightarrow \quad a = \frac{F}{m}\,, \quad \text{and} \quad m = \rho A l \quad \rightarrow \quad a = \frac{\Delta P}{\rho l}\,.$$

Note that the area A canceled out.

That means that, *if* the pressure on the air stayed the same at all times, the velocity of the air would become,

$$v = \frac{\Delta P}{\rho l} t\,.$$

The pressure inside the bottle falls because air is leaving the bottle. The volume of air which leaves the bottle is the width of the neck times how far the air in the neck moves. How far the air moves, is its velocity times time. This gives, *roughly*,

$$\Delta V = A v t = \frac{A \Delta P}{\rho l} t^2\,.$$

Now, how much air needs to leave the bottle for the pressure to fall down to atmospheric? That will tell me what t has to be, for the pressure to go from maximum to atmospheric. The pressure loss inside the bottle is atmospheric pressure times the fraction of the air I take out. That is, if I take out 1/10 of the air in the bottle, the pressure will fall by 1/10 of an atmosphere, since it is the number of air molecules in the bottle which determines how high the pressure is. Therefore, to get rid of the overpressure, I need,

$$\frac{\Delta V}{V} = \frac{\Delta P}{P_{\rm atmos}}\,,$$

which gives,

$$\frac{A \Delta P \; t^2}{\rho l V} = \frac{\Delta P}{P_{\rm atmos}} \quad \Rightarrow \quad t^2 = \frac{\rho l V}{A \Delta P} \, \frac{\Delta P}{P_{\rm atmos}} = \frac{\rho l V}{A P_{\rm atmos}}\,.$$

Recall from long ago, that $P_{\rm atmos}/\rho = v^2_{\rm sound}$, the square of the speed of sound.[1] Therefore, I can re-write this all as,

$$t^2 = \frac{l V}{A v^2_{\rm sound}}\,.$$

[1] Actually this is not quite right; there was a tiny correction $\sqrt{c_p/c_v}$. However, the exact same correction actually should have come up here. For the physicists, what is important in both cases is actually dP/dV the change in pressure with volume. Since this is the right quantity in both cases, the derivation I made above is actually wrong until the moment I make this substitution, and then it becomes correct!

This is the time for one quarter of the resonance phenomenon to go by. Therefore, my rough estimate is that the whole resonance phenomenon will have a period which is 4 times this long,

$$T = 4\sqrt{\frac{lV}{Av_{\rm sound}^2}} \, .$$

However, this is an under-estimate. The reason is, that I assumed that the full pressure was acting on the air in the neck the whole time; really, once the air starts to move, the pressure is falling. Also, I assumed that the whole velocity was present from the start; really, at first the velocity is zero, and it only builds up with time. Therefore, the real answer will be a little longer than this. You should not be surprised to learn that the right answer is actually,

$$\boxed{T = \frac{2\pi}{v_{\rm sound}}\sqrt{\frac{lV}{A}} \qquad \text{or} \qquad f = \frac{v_{\rm sound}}{2\pi}\sqrt{\frac{A}{lV}} \, .}$$

Now think about this logically. Does each term in the expression belong there?

- Dependence on A: if the neck is wider, the air can empty out of the bottle faster. That means the pressure falls faster, and the whole process does not take as long. Larger A gives smaller T, or larger f. That is correct.

- Dependence on l: if the neck is longer, the air in the bottle is pressing on a bigger mass of air. It takes longer for this larger amount of air to get moving. That means the whole thing occurs more slowly. Larger l then gives larger T and smaller f. That is correct.

- Dependence on V: if the volume is bigger, more air has to leave the bottle before the pressure falls. That takes longer, meaning a larger T or smaller f. That is correct.

- Dependence on $v_{\rm sound}$: faster sound speed means air responds faster to its environment, meaning smaller T and larger f. Again, that is the behavior we found in our "derivation."

The bottle we considered is one of the simplest systems one can consider, to find a resonant frequency. For something more complicated, such as the air cavity in the mouth or hands when you whistle or hand-whistle, it is not so easy to do the calculation. There are a few other cavities simple enough that the resonant frequency can be found in closed form, and it turns out that the standard wind musical instruments are each quite close in shape to one of those other cavitites. For instance, a clarinet's bore (the hollow space inside the wood of the instrument) is almost of constant diameter and is almost completely closed at the reed and open at the opening: so it is pretty similar to,

a cylindrical tube. We can make a feeble attempt to use the same formula we found above to describe this tube. The problem is that the whole volume is inside the neck. Still, throwing caution to the winds and admitting that we do not expect the right answer, I can identify the volume of the bottle with the volume of the clarinet and the length of the bottle neck with the length of the clarinet; so $V = Al$. In this case, the formula is $f = (v_{\rm sound}/2\pi)\sqrt{A/l \times Al}$, or $f = v_{\rm sound}/2\pi\, l$. The right answer turns out to be, that you should replace 2π in this expression with 4:

$$f_{\rm cylinder} = \frac{v_{\rm sound}}{4\, l}\,.$$

This explains why your 2.5 cm meatus enhances sounds with a frequency of 3500 Hertz: $f = v_{\rm sound}/4\, l \simeq 3400$ Hz.

Non-science people can definitely skip reading the following.

[There is actually a general procedure for finding the resonant frequencies of any cavity with a narrow opening, but it requires differential equations. One assumes that the pressure is changing sinusoidally. The relations between pressure and air velocity are,

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \frac{1}{\rho}\vec{\nabla} P\,, \\ \frac{dP}{dt} &= \frac{c_p P_{\rm atmos}}{c_v}\vec{\nabla}\cdot\vec{v}\,, \end{aligned}$$

from which it follows that,

$$\frac{d^2P}{dt^2} = \frac{c_p P_{\rm atmos}}{c_v \rho}\nabla^2 P = v_{\rm sound}^2 \nabla^2 P\,.$$

Look for solutions which are sinusoidal,

$$\nabla^2 P = -\frac{\omega^2}{v_{\rm sound}^2} P\,,$$

with boundary conditions of $P = 0$ at the openings. Typically solutions will exist for several discrete choices of ω, which give the resonant frequencies. The reason for the boundary conditions is, that air can escape efficiently from the opening, so it only requires a tiny pressure to allow an airflow at this point. Beyond the narrow opening approximation, life becomes difficult because the resonances are damped and do not have precisely defined frequencies.]