Lecture 20: Pipes, resonances, standing waves

Very few instruments resemble the bottle we considered last time. (Your mouth when you whistle and your hands when you hand whistle, and those simple clay whistles with several finger holes and police whistles are examples of a few which do.) Most wind instruments which appear in orchestras come fairly close to being described as:

- A cylinder with one open and one closed end: the clarinet, some organ pipes and pan pipes (traditional instruments played, for instance, by Peruvians), the tubes underneath a xylophone, etc.
- A cylinder, open on both ends: the flute, some organ pipes, etc.
- A cone: oboe, bassoon, saxophone, some organ pipes, etc.

The brass instruments are more complicated; part of the tube is cylindrical, but part opens out, but not like a cone. We will return to that when we discuss brass instruments.

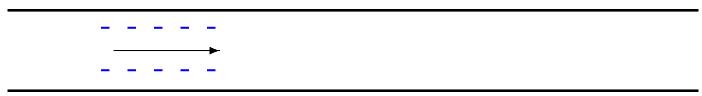
We need to think about how to describe the resonances in these instruments. It is also very important, not *just* to determine the lowest resonant frequency, since the tone quality and useful range of these instruments comes about because of the resonances higher than the fundamental.

To start out, think about a tube, open on both ends. Imagine that for some reason, a sound wave is moving down the tube, close to one end:

Obviously it moves to the far end of the tube,

where it reflects. Because the tube is opening out (easy for air to flow, but you cannot put up a net pressure), the reflected wave has the same air motion but opposite pressure:

This wave is moving back the other way, and goes towards the beginning of the tube:



Again it will reflect; again the pressure will switch sign but the air motion will not:

We are right back where we started. It is also simple to compute the time it takes for this process to occur: it is the time it takes for the air to go the length of the tube, reflect, and come back. That is the time it takes sound to go *twice* the length of the tube:

$$\boxed{\text{Open-open tube: } T = \frac{2L}{v_{\rm sound}}\,, \qquad f = \frac{v_{\rm sound}}{2L}}$$

for the wave to come back to what it was at the beginning.

We can ask the same thing about a tube, closed on both ends. The difference is that, on each reflection, the pressure will be the same but the air velocity will switch (remember, no air can flow out the end of a closed tube!). Again, the sound wave will look the same after making a full transit of the tube, and

$$\boxed{\text{Closed-closed tube: } T = \frac{2L}{v_{\rm sound}}\,, \qquad f = \frac{v_{\rm sound}}{2L}}$$

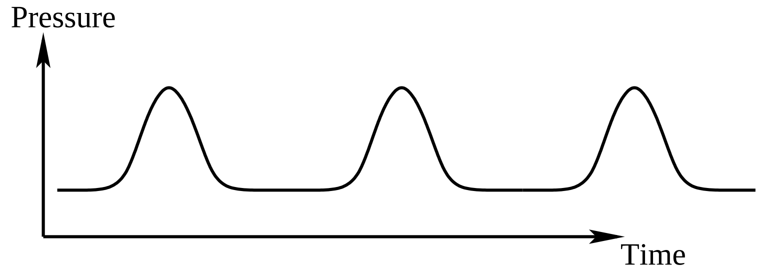
What about a tube, open on one end but closed on the other? That is more subtle, because the pressure flips sign on one reflection (from the closed end), but not on the other. Therefore, the pictures in that case will look like,

- positive pressure wave goes from beginning to end, reflects off open end
- positive pressure wave goes from end to beginning, reflects off closed end
- negative pressure wave goes from beginning to end, reflects off open end
- negative pressure wave goes from end to beginning, reflects off closed end

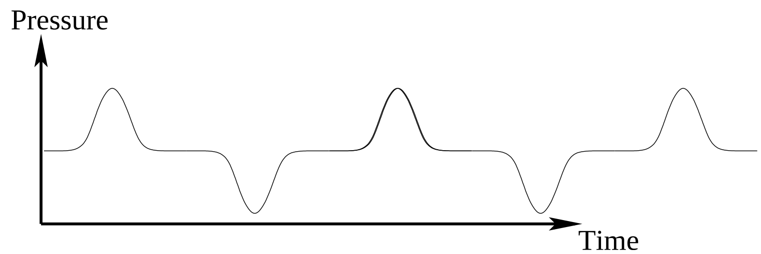
which means that it takes 4 transits of the tube for the situation to return to what it was:

$$\boxed{\text{Open-closed tube: } T = \frac{4L}{v_{\rm sound}}\,, \qquad f = \frac{v_{\rm sound}}{4L}}$$

What will be the appearance of the sound wave emitted from the right hand end of the tube? There will be a pressure pulse each time the sound wave bounces off that end. For the open-open tube, it is always positive pressure approaching, and we get,



while for the open-closed tube, the sound wave alternately approaches with positive and negative pressure (relative to atmospheric), and so the sound wave looks like,



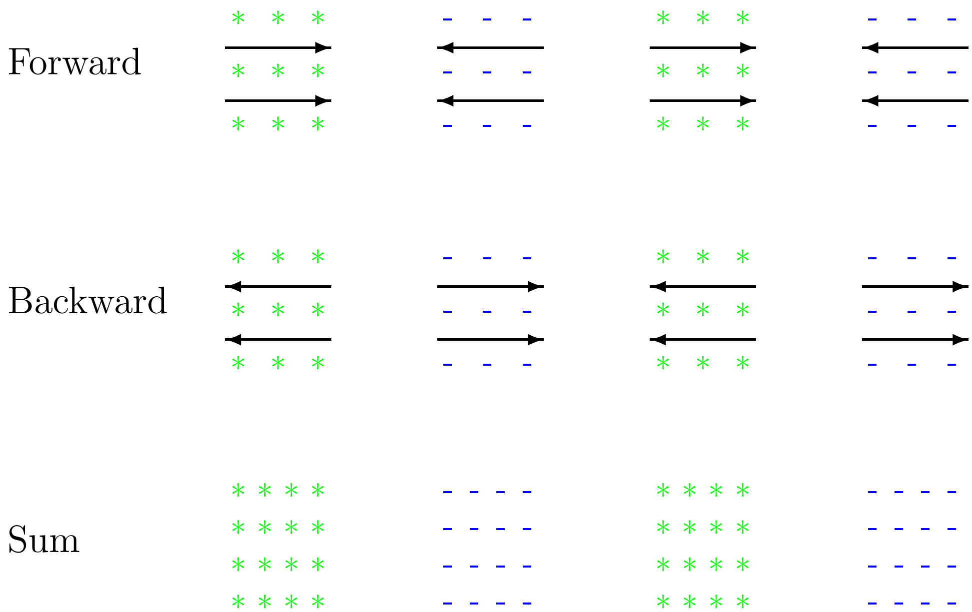
Neither of these is a sine wave, so what I have been talking about is actually having a bunch of sine waves in the tube at once. However, each is periodic, which shows that the sine waves in the tube will have frequencies which are *integer multiples of* (harmonics of) a fundamental frequency which is what I have given above. This is a neat property of such tubes, and is the reason that they are used musically.

Now we have to try harder to figure out exactly what harmonics are possible and what pattern they make inside the tube. This is important to the way musical instruments will function, so we will go through it in a little detail.

A sound wave of a given frequency will be a sine wave. Compared to what we were just considering, that means there will be a wave traveling in each direction along the tube at all times. To understand what will happen, we should think about what it looks like when two sine wave sound waves are going in opposite directions to each other. This can be inside of a tube or in the open air, it doesn't matter too much–the point is to understand what the pattern of pressure and air flow looks like when there are overlapping waves moving in opposite directions. When there are two waves moving in opposite directions and of equal strength, this is called a **standing wave**.

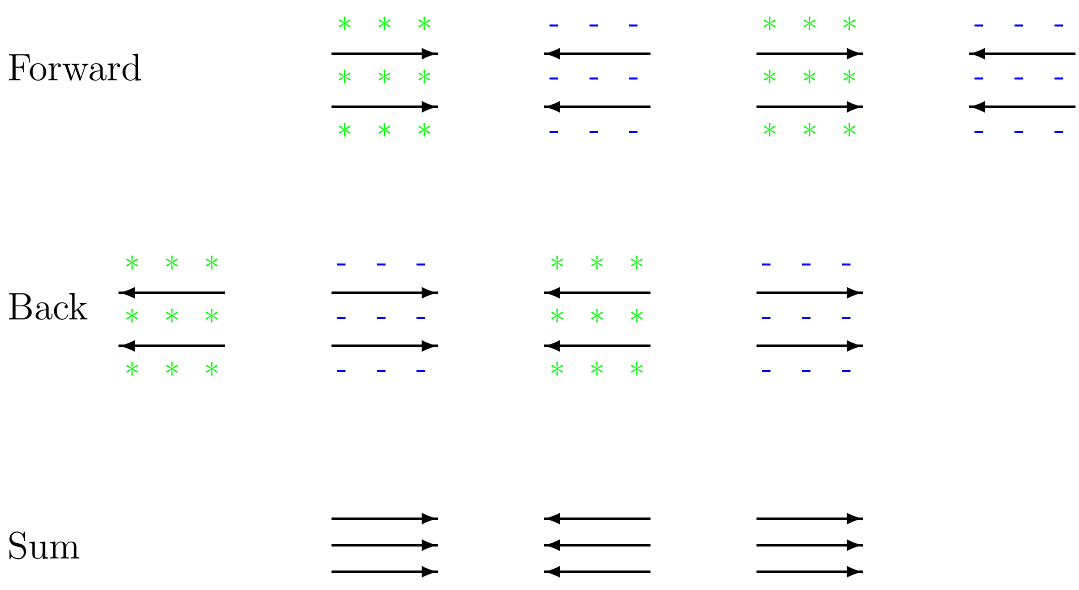
Consider two waves of equal wavelength and equal intensity, going in opposite directions (a standing wave). The two waves coincide in space. The total pressure and velocity of the air is the sum of the contributions from the two waves. To draw what happens, it is convenient to draw the pressure and velocity pattern of each wave, one above the other, and then draw what will be the sum of those two.

At the moment when the pressure peaks are at the same points in space, this looks like,



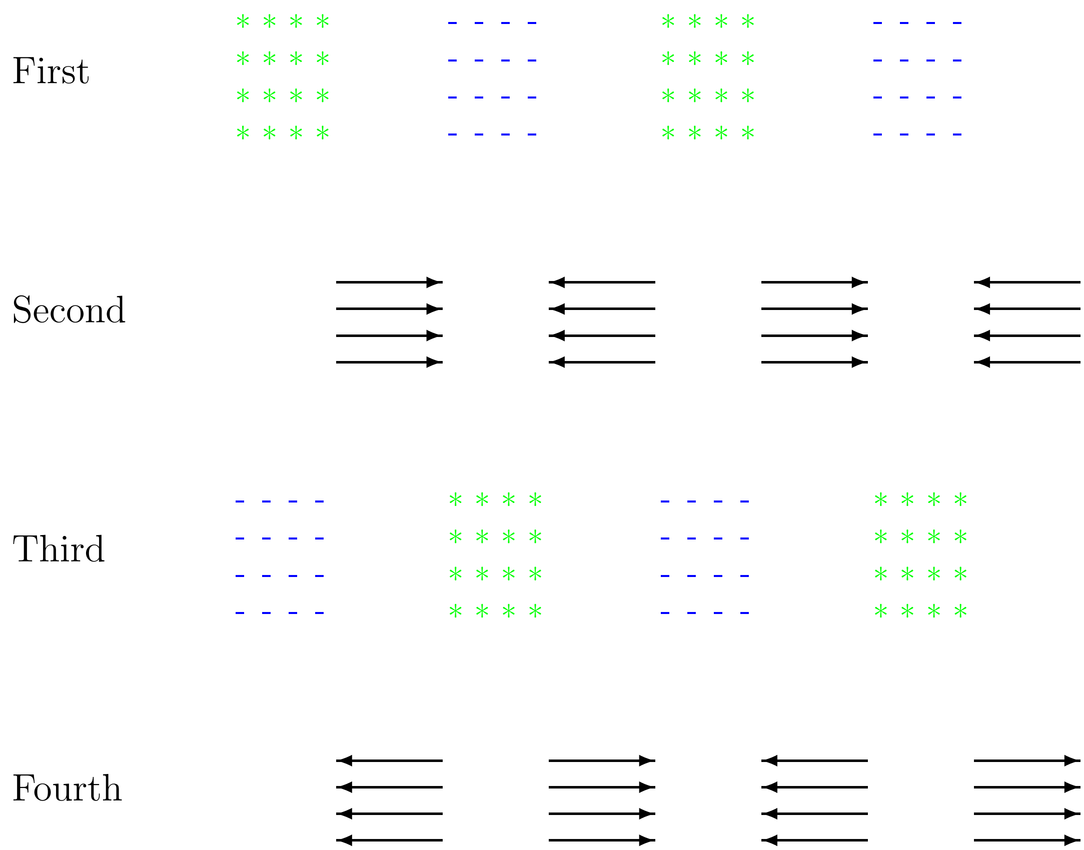
The air motions cancel, but the pressures add. There is a pattern of alternating high and low pressure.

A moment later, when the forward wave has gone forward a quarter wave length and the backward wave has gone backward a quarter wave length, it looks like this:



Two things have changed. First, now the pressures cancel and the air motions add. Second, the location where this happens is different: the spot where the air speeds are maximum is in between the spots where the pressure peaks occurred. The air speed maxima occur 1/4 of a wavelength off from where the pressure peaks occur.

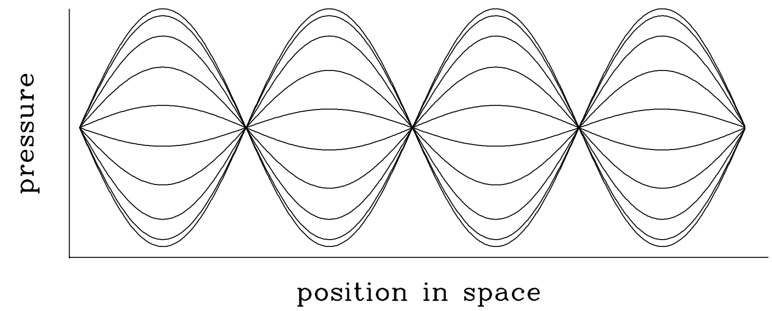
The time series of how a standing wave works, therefore looks like this:



Why? Let's go from "First" to "Second": the area with a high pressure behind it and low pressure in front of it is getting pushed forward. Therefore the air here starts to move forward. The area with low pressure behind and high pressure in front gets pushed backward. The reason that it is the spots *between* pressure peak and trough which start to move, is that those are the places where the force in front and behind are different from each other.

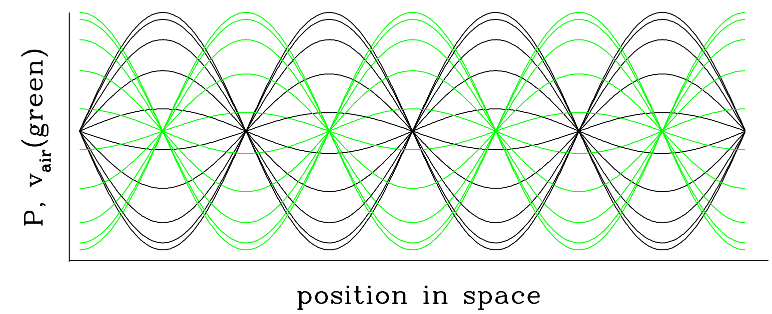
Now go from "Second" to "Third." The spot where the air is going forward from behind and backward from in front, is getting compressed (since air is rushing into that spot). It will therefore become a high pressure region. The spot where air is moving backwards behind and forward in front will become rarefied (less air), and therefore low pressure. The pressure pattern is the reverse of the starting one, so the last two steps are just the reverse of the first two.

If we make a plot of the pressure against position, and superimpose the curves for many different times, it looks like,



The spots where the pressure stays constant are called **pressure nodes**, and the places where it changes the most are called **pressure antinodes**.

Drawing the air motion in green, and putting the two plots on top of each other, the behavior of pressure and of air motion looks like,



Note that, as explained before, the **velocity nodes** (where air velocity stays constant) are at the same place as the *pressure antinodes*, while the **velocity antinodes** (where velocity changes the most) are at the same places as the *pressure nodes.*

Inside a cylindrical tube, a sound wave of definite frequency will look like a standing wave of definite wave length. Note that

the wave length is twice the distance between pressure nodes.

All we have to do is figure out what should happen at the ends of the cylinder.

Air cannot go out through a closed end of a tube, but the pressure can be whatever it needs to be there. Therefore,

Closed ends are velocity nodes and pressure antinodes.

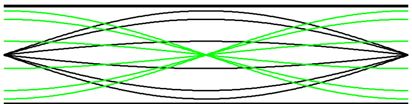
At the open end of a tube, air is free to move in and out. However, the pressure cannot differ from atmospheric (think about what happens when a sound wave in a narrow tube reflects from a widening, and think of the end of the tube as the tube becoming extremely wide). Therefore,

Open ends are velocity antinodes and pressure nodes.

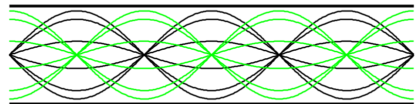
Now we are ready to think about the set of modes in each kind of cylinder. The lowest frequency standing wave that can occur in a cavity (such as a cylinder) is called the **fun-**

damental resonant frequency of the cavity, and the higher ones are called **overtones** of the cavity.

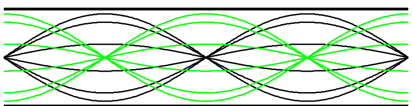
Consider first a cylinder open on both ends. The ends have to be pressure nodes. Therefore, the first few resonant patterns possible in the cylinder are,



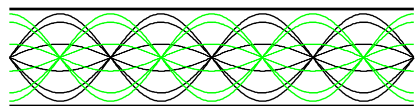
Fundamental



Second overtone: 3×



First overtone: 2×

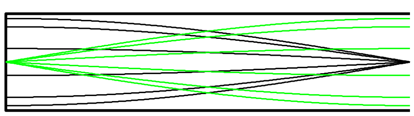


Third overtone: 4×

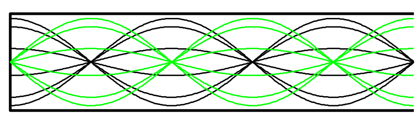
which are respectively, fitting 1/2 wave length, 1 wave length, 3/2 wavelengths, and 2 wavelengths of sound into the tube. Therefore they have wavelengths of $2L$, $2L/2$, $2L/3$, $2L/4$, *etc.* and frequencies,

$$\boxed{\text{Open-open tube, length } L: \quad f = \frac{v_{\rm sound}}{2L} \times \Big(1\,,\,2\,,\,3\,,\,4\,,\,5\,,\,\ldots\Big)}$$

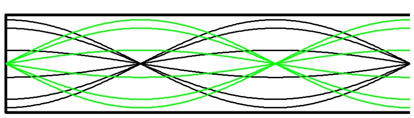
If instead I consider a tube which is closed on one end and open on the other, there should be a velocity node on the closed end (and therefore, a pressure antinode), and a pressure node (velocity antinode) on the other. The first few resonant patterns fitting the bill are,



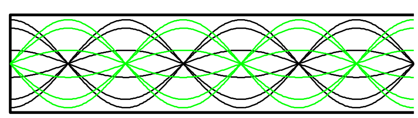
Fundamental



Second overtone: 5×



First overtone: 3×



Third overtone: 7×

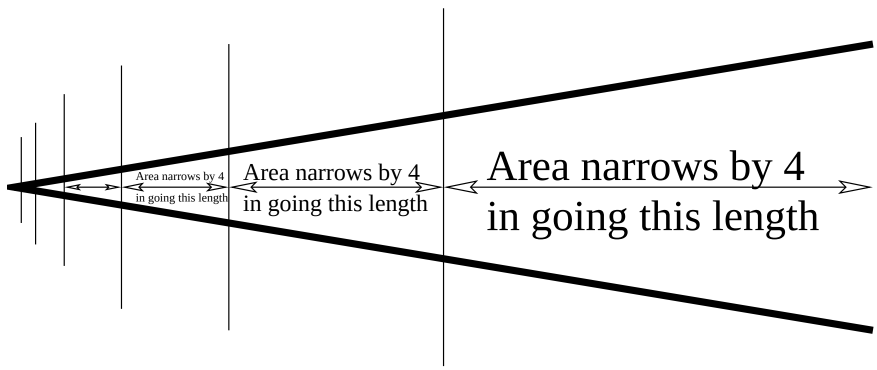
which are 1/4 of a wavelength, 3/4 of a wavelength, 5/4 of a wavelength, and 7/4 of a wavelength fitting inside the tube. Therefore the resonances have wavelengths of $4L$, $4L/3$, $4L/5$, $4L/7$, *etc.* and frequencies,

$$\boxed{\text{Closed-open tube, length } L: \quad f = \frac{v_{\rm sound}}{4L} \times \Big(1\,,\,3\,,\,5\,,\,7\,,\,\ldots\Big)}$$

This has half the frequency as a fundamental, and is missing the even multiples of the fundamental.

What about a conical tube? This is an important problem in music, because several instruments are shaped approximately conically, including the double reed instruments and the saxophone.

The problem with analyzing the conical tube is that we can no longer just say that a sound wave will propagate along the tube without any change. Recall that, if the width of a tube is changing slowly, a sound wave propagates without reflection. If the area of the tube changes by a large ratio in much less than the wavelength of the sound, though, then most of the sound power is reflected. Now think about a cone:



Any wave will reflect at some point. The longer the wavelength, the further out in the cone it will reflect. Therefore, the cone is closed-open, but it acts like it is shorter for the low modes, and closer to its full length for the high modes. The resonant frequencies turn out to be exactly those for an open-open tube:

$$\boxed{\text{Conical tube, length } L: \qquad f = \frac{v_{\rm sound}}{2L} \times \Big(1\,,\,2\,,\,3\,,\,4\,,\,5\,,\,\ldots\Big)}$$

but the pressure pattern is different; in stead of the pressure coming to zero at the tip, it has a node there.

[For those with advanced mathematical background, the explanation is that finding the pressure pattern inside the cone is solving the wave equation in *spherical* coordinates. We want pressure to satisfy,

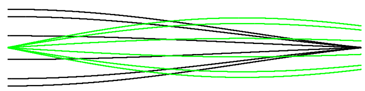
$$\nabla^2 P = \frac{-\omega^2}{v_{\rm sound}^2} P$$

with $P = 0$ at the opening and $dP/dr = 0$ at the tip. Work in spherical coordinates with the tip of the cone as the center of the sphere. The pressure will depend on r only. The solutions are spherical Bessel functions. In particular, the pressure is given by

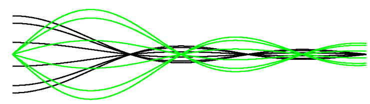
$$P(r) = P_0 \frac{\sin(kr)}{r}\,, \qquad k = \frac{2\pi}{\lambda}$$

which has zeros exactly where sin has them, except with the zero at the origin removed. The zeros are at the same places as for $\sin(kr)$, the function we need for the open-open pipe.]

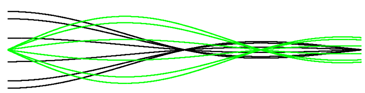
The patterns of pressure and air speed in the conical pipe for the lowest modes, not bothering to draw the cone this time, look like,



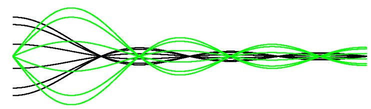
Fundamental



Second overtone: 3×



First overtone: 2×



Third overtone: 4×

These figures are slightly deceptive in terms of showing where the energy in the wave is stored. The pressure and velocity tell what the intensity is, and it is clear that the wave is most intense near the tip of the cone. However, the most *power* is further out, because the power is the intensity times the cross-sectional area of the cone, which becomes small near the tip. Therefore, most of the sound energy does not go back into the tip of the cone.