Lecture 21: Decay of Resonances

Recall that a resonance involves energy bouncing back and forth between two means of storage. In air resonances in cavities, energy is going back and forth between being stored in air compression, and being stored as air motion. However, in any resonant process in the real world, a little of the energy will go into other things each time it goes back and forth between the two main storage mechanisms. For instance,

- when you swing on a swing, energy is going back and forth between gravitational potential (being high up at the top of the swing) and energy of motion. However, there is energy being lost to friction against the air, energy going into the bar or branch holding up the swing, energy lost as friction in the joint or pivot on which the rope or chain swings, and so forth. Therefore, unless you keep pumping the swing, it will slowly go less and less high.

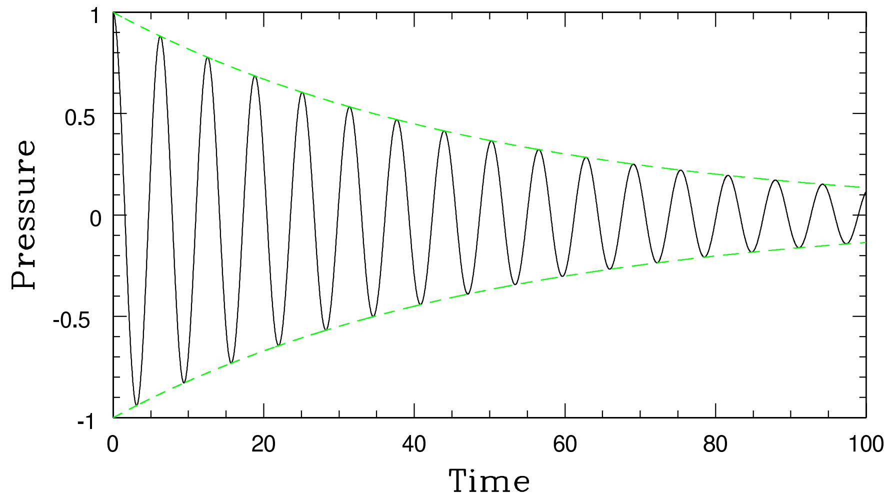
- When sound is resonating inside of a wind instrument, some sound energy is escaping from the instrument into the world through the fingerholes, opening, or walls of the instrument. Some (typically much more) is being lost to friction as the air rubs against the walls of the instrument. Some may be lost to inelasticity in the reed, lips, and so forth.

- In a string instrument or piano, the energy in the vibration of the string is lost through the bridge of the instrument into the instrument's body, and against the soft pad of the finger.

- In percussion instruments, energy is lost to radiated sound, but also to inelasticity of elements (especially wood components, drumheads, and other soft pieces) and is carried out through the support structure. It is also removed by deliberate damping devices.

Just how much of the energy is lost per cycle of the resonance is important in understanding how an instrument built around resonances will behave. Therefore we will try to quantify resonant energy loss in this lecture. We will also try to understand what happens when you **drive** a resonance, which means, when you "push on" something which can behave resonantly, at a frequency which is close to the resonant frequency. This explains how an instrument built around a resonating device can produce a steady tone which does not die away with time.

First, some terminology. The behavior of one variable in a resonance which is dying away typically looks like this:

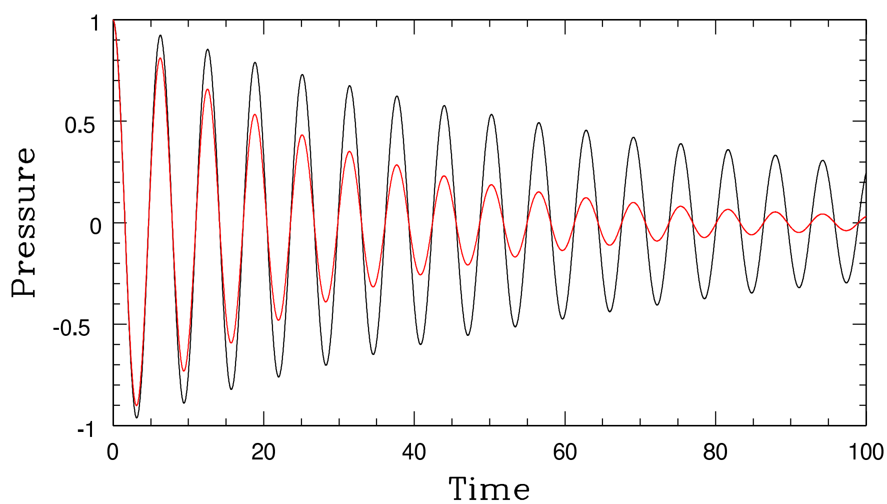


I took the vertical axis to be pressure for the sake of argument; it could be air velocity, your forward speed on a swing, or whatever. The green dotted lines are called the **envelope** of the decaying oscillations.

The curve in the figure is characterized by three things:[1]

- The period, or equivalently, the frequency, of the oscillations. The signal is no longer sinusoidal, but we can define the period as the time between peaks in the pressure.
- The starting size of the pressure oscillations.
- The rate at which the oscillations damp down.

The first two of these we are familiar with. We need some language to describe the last one. For instance, the following two curves have the same period and starting pressure:



[1]There is really a fourth, the starting phase of the oscillations, but it is not really interesting.

How do we describe that the black curve takes a long time to die away, and the red curve dies away faster?

Resonance is a transfer of energy between two things: air pressure to air motion to air pressure to air motion to air pressure .... The relevant question is, what fraction of the energy is lost in each transfer from one form to the other? Since we like resonances which take a long time to die away, we define the **resonant quality** Q as, roughly,

$$Q \sim \frac{\text{Energy in resonance}}{\text{Energy loss per transfer}}$$

where by transfer I mean transfer of the energy from one form of storage to the other. It turns out to be more convenient to define Q as precisely,

$$Q \equiv 2\pi \, \frac{\text{Energy in resonance}}{\text{Energy loss in 1 period}} \, .$$

The 2π is the usual 2π that comes into anything involving sines and cosines.

A large Q means the resonance repeats for many oscillations without losing strength. A small Q means that it only lasts a few.

An equivalent way of writing Q is,

$$Q = 2\pi f \frac{\text{Energy}}{\text{Power loss rate}}$$

[The reason for the 2π in the definition is that $2\pi f = \omega$ the angular frequency, which turns out to be a more natural quantity physically.] Note that it could sometimes be true that a bigger or smaller fraction of energy is lost from the early oscillations than the late ones. However, for *most* resonant phenomena, this is not true; if there is twice as much energy in the resonance, twice as much will be lost.

So how long does it take for a sound to die away? Very roughly, the answer must be $Q/2\pi$ oscillations. However, each oscillation starts with less energy, and so each oscillation loses less energy than the one before it. Therefore, the energy loss is actually exponential:

$$\frac{\text{Energy}}{\text{starting Energy}} = e^{-2\pi \times (\text{number of oscillations})/Q} \, .$$

[Here $e = 2.71828$ is the base of the natural logarithm. If you don't know how that got in here, don't worry about it, just take this as a result.]

How long it takes for a resonance to "die away" depends on what your standards for "die away" are. Technically, the resonance will go on forever, just smaller and smaller. However, at some point it is so small that it has disappeared for practical purposes. Therefore, we will define a **decay time** τ, as the time for the amount of energy to fall by a factor of $1/e$. This is,

$$\frac{1}{e} = e^{-2\pi(\text{number of oscill})/Q}$$

or,

$$1 = \frac{2\pi(\text{number of oscillations})}{Q}$$

or,

$$\text{number of oscillations} = \frac{Q}{2\pi}\,,$$

but the time is the number of oscillations over the period, which is the number of oscillations over the frequency. Therefore,

$$\tau = \frac{Q}{2\pi f}\,,$$

is the decay time.

The ear considers a change in intensity by a factor of 10 to be roughly a factor of 2 change in loudness, as we have seen previously. The time for a factor of 10 loss in intensity is

$$\tau \log_e(10)$$

and for a factor of 100 loss is $\tau \log_e(100)$, and so forth. Roughly,

$$\log_e(10) = 2.3\,.$$

For example, listening to the diedown in the ringing made when I "pluck" the wine bottle with my thumb, I estimate it takes 1/3 of a second for the sound to die away. That gives me $\tau \sim 0.3$ s$/2.3 = 0.13$ seconds. The frequency is 112 Hertz, so the Q is $Q = 2\pi f\tau = 6.28 \times 112 \times .13 = 90$. This is considered a reasonable Q but not super high.

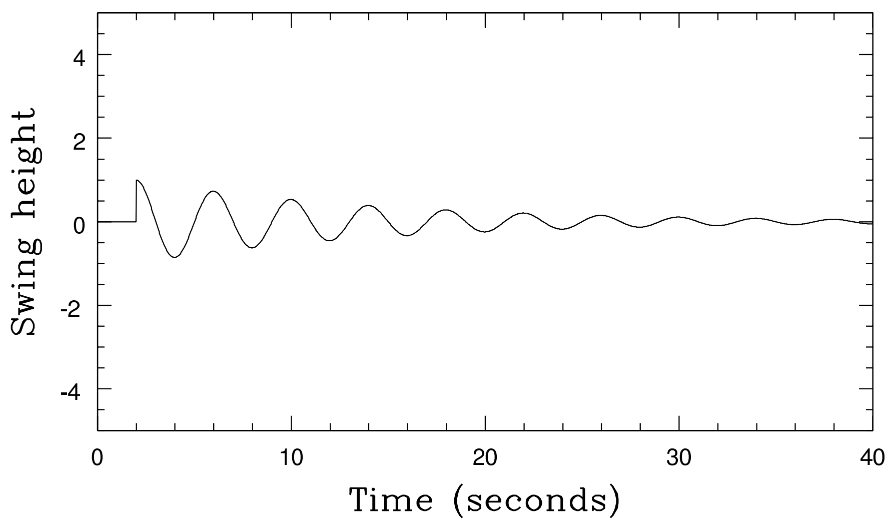
I can try to do a better job by actually looking at the pressure pattern of a recording of "plucking" the bottle. I count how many pressure peaks for the peaks to get smaller by a factor of 2. Since intensity goes as pressure squared, this is a change in intensity of a factor of 4, requiring $\log_e(4) = 1.4$ times the decay time. I count about 16 oscillations for the amplitude to fall by this factor of 2. Therefore the τ is $16/(1.4 \times 112)$ seconds, which is .10 seconds, about the same as my "by-ear" estimate. The Q can be found directly from the number of oscillations for the energy to fall by some factor:

$$Q = \frac{2\pi \times \text{ number of oscillations}}{\log_e(\text{start/final energy})}$$

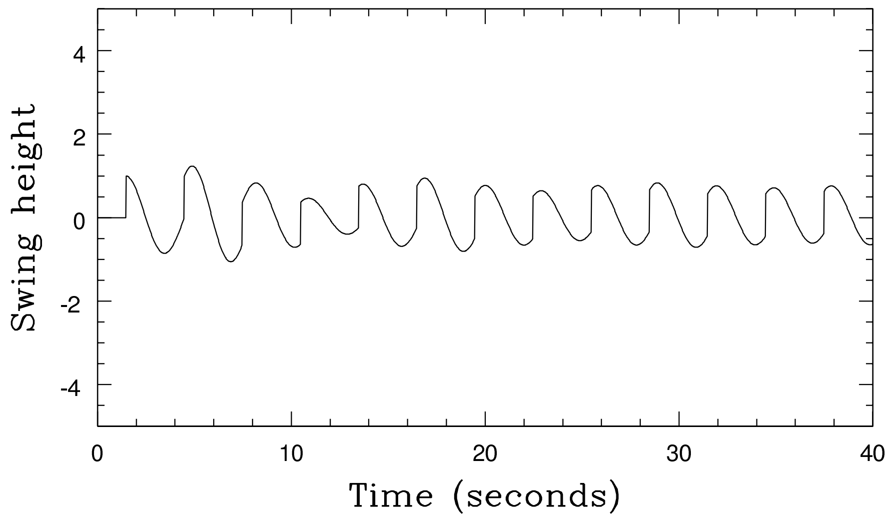
which gives $2\pi \times 16/\log_e(4) = 72$ for the Q, about the same as the other estimate. Doing the same thing with the open D string on the cello, I find about 60 oscillations for the pressure peak to fall by a factor of 2, so $Q \sim 270$. The Q of instruments are often in this range of 100 to hundreds. Some metal percussion instruments have much higher Q.

For an instrument to make a steady tone, it has to have energy go in as fast as it is going out from the resonance. I have to **drive** the resonance somehow. Resonances are also useful if something else happens to be producing periodic sound at a frequency close to the resonant one. The size of the resonant oscillations then depends not only on how loud the other sound is, but on how close its frequency matches the resonant frequency. This will explain some things we have already discussed, about the ear and about other things, so let us see it in a little detail.

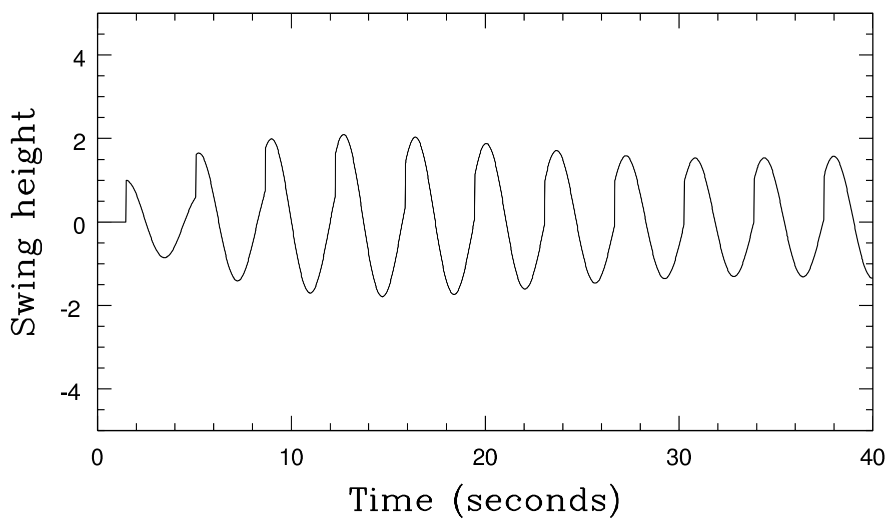
Think about pushing your little brother or sister on a swing. Suppose the swing goes back and forth in 4 seconds. If you push it just once, when it is at the bottom, the swing will do this:



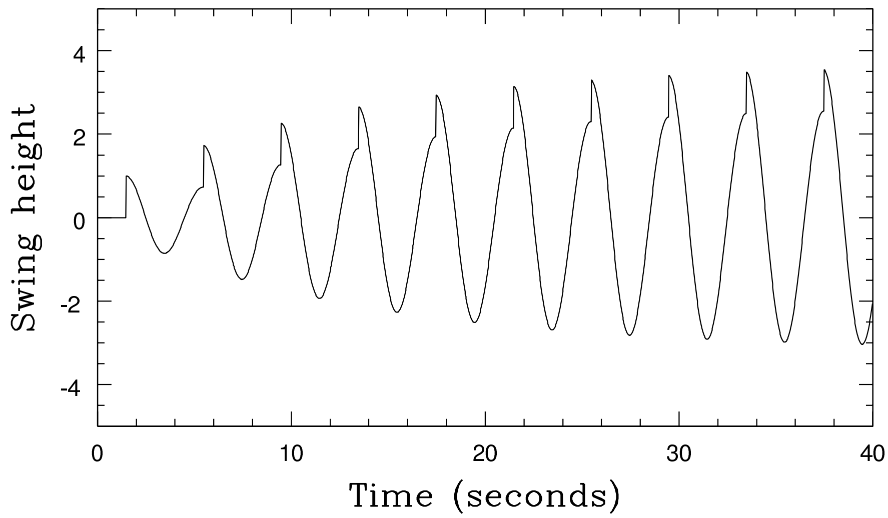
Suppose you push the swing every 3 seconds, which is too often, as you know. It means you will be pushing the swing before it reaches the bottom from the last swinging. The swing will do this:



Because you are not pushing the swing when it is going forward, you don't really get a larger amplitude swinging than the result of each push. What if you push closer to the right frequency, say, every 3.6 seconds?



That worked much better. However, it does not work perfectly, because you are pushing too soon; the swing has not come back to where the push will do the most good. The way to think about it is that the last push you made was .4 seconds too soon; the push before was .8 seconds too soon; going back 5 pushes, it was 2 seconds too soon. That means that the swinging because of that push, is backwards compared to what you are doing now. Only the last 5 pushes are actually adding up. (The reason it is 5 pushes is, that your are pushing 0.4 seconds too early, and it takes 5 pushes for that to add up to half a swing.) If you push every 4 seconds, the ideal frequency, you would get this:



Now each push is coming exactly when it is needed. This leads to the maximum height for the swing. The only reason the swinging does not grow bigger and bigger is that the swing is damped.

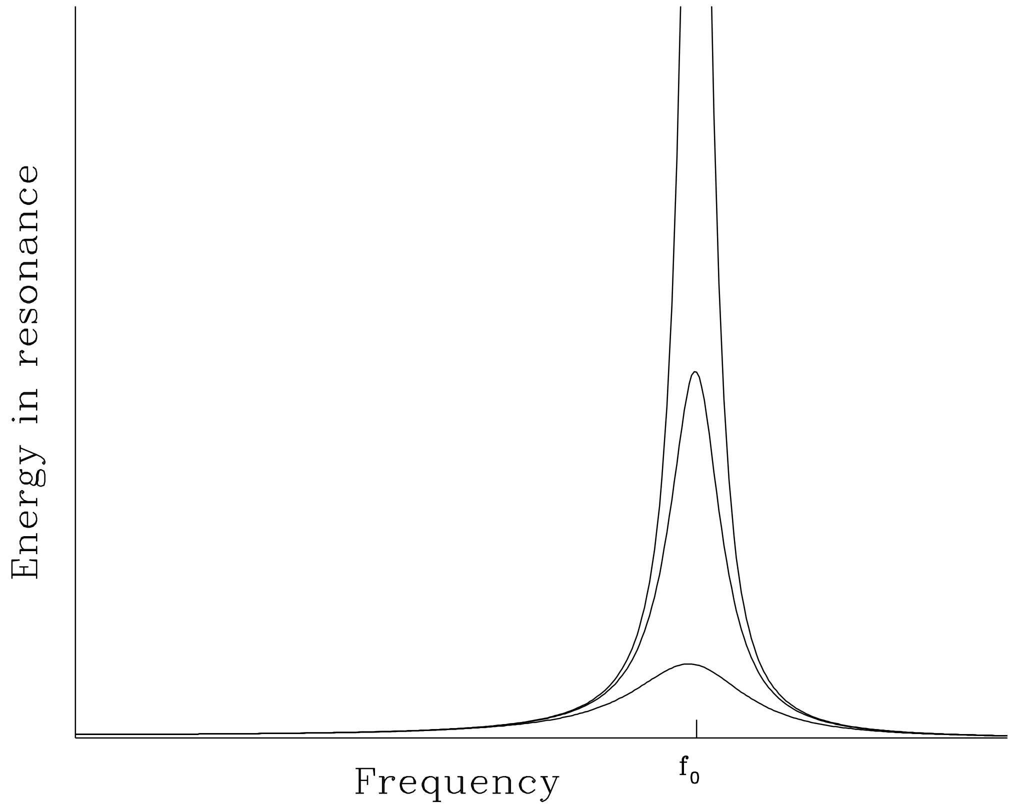
How big can the amplitude get, and how close to the right frequency must the pushes come? I will answer for the case where you push with a sine wave, instead of discrete, separate shoves. Call the ideal frequency f_0, and the frequency you really push with, f. If $|f - f_0| > f_0 Q/2$, then the height is limited by the fact you get out of sync with the result of previous pushes. The amplitude then grows with the accuracy that you match your pushes:

$$\text{Amplidude} \propto \frac{f_0^2}{(f_0{+}f)|f_0{-}f|}$$

(Here $||$ is absoluted values; if the thing inside is negative, choose minus that thing. $\propto$ means "is proportional to.") If the frequencies are better matched than this, the Q is the thing that limits how big the oscillation gets, and

$$\text{Amplitude} \propto \frac{1}{Q}\,.$$

The energy in the resonance goes as the square of the amplitude. Therefore, a forced resonance, with the forcing at the right frequency, has its energy grow as $1/Q^2$. The energy in the resonance, as a function of frequency for a few Q values, is shown below:



The moral is, that the way to get a big amplitude vibration is to have a high Q and drive on resonance.

For a musical instrument, the resonance is typically at hundreds of Hertz. There is no way a person can “push the swing” at just the right time, at hundreds of Hertz. Therefore there has to be an *automatic* way that energy gets fed in at just the right spot in the resonant cycle. This is one of the three *key ideas* of designing a steady-tone musical instrument:

Key ideas in steady tone instruments:

- Have a high Q resonance.
- Have a way (nonlinearity) that energy is automatically fed in at the right point in the resonant cycle.
- Have some way to convert the energy in the resonance efficiently into sound waves.

We will see how various wind and string instruments solve these problems in the coming lectures.