

Lecture 7: Superposition and Fourier Theorem

Sound is **linear**. What that means is, if several things are producing sounds at once, then the pressure of the air, due to the several things, will be

$$P_{\text{air}} = P_{\text{atmospheric}} + P_{\text{source 1}} + P_{\text{source 2}} + P_{\text{source 3}} + \dots$$

and the air velocity will be

$$v_{\text{air}} = v_{\text{wind}} + v_{\text{source 1}} + v_{\text{source 2}} + v_{\text{source 3}} + \dots$$

How each sound wave propagates, is independent of what all the other ones do. Therefore it is fully legitimate to solve for the behavior of the sound wave for each source, one at a time, and add them up. Further, if there is a sensible way of “breaking up” the sound wave from *one* source into pieces, you can treat each piece separately and then add them back together.

These statements are true so long as

$$v_{\text{air}} \ll v_{\text{sound}} = 340 \text{ m/s}$$

and

$$P_{\text{air}} - P_{\text{atmospheric}} \ll P_{\text{atmospheric}}$$

that is, so long as the velocity and pressure being caused by the sound wave are small compared to the speed of sound and ambient pressure. Even for a sound at the threshold of pain, these conditions hold by a cool factor of 10 000. [They break down at about 200 decibels.] Only violent explosions would violate these conditions, so we will always assume they are true in this class.

What linearity means is that a complex sound—say, an orchestra—can be described as the **sum** of simple sounds—say, each instrument. Sometimes, your ear can make the separation between these sounds (though not always, as we discuss when we talk about masking).

We have already seen that musical sounds, at least the ones of definite pitch, are periodic. But they are *not* generally sinusoidal. This matters because everything we said last lecture, about how a sound causes vibration on the cochlea, is true of sine wave sounds. The purpose of this lecture is to start seeing why that understanding is still good enough to describe complicated sounds, and what goes on when a complicated sound enters your ear.

The sound files accompanying the lecture (in the HTML version) play a sine wave, triangle wave, square wave, and sawtooth wave, as well as a voice “singing” (quotes, because it was me) “ooo” and “eee”. The wave forms of “ooo” and “eee” are shown in Figure 4 at the end of these notes. The computer generated waves have the same frequency, 440 Hertz, and the sung notes have the same frequency, 256 Hertz. They sound very different, that is, they have different **timbre**.

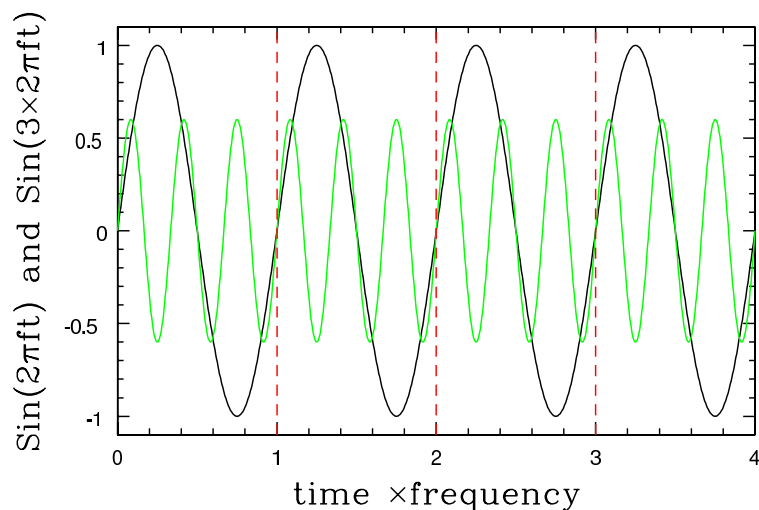


Figure 1: Four periods of a wave with a certain frequency, and a wave with exactly 3 times the frequency. Both waves certainly repeat at each point where the red line is drawn.

- Pitch: determined by a periodic sound’s frequency.
- Timbre: determined by the “shape” of one period of the wave.

Can we describe timbre more precisely? Yes. That is because of a powerful piece of mathematics, called **Fourier analysis**, which says that any periodic function—say, the pressure pattern from an instrument—can be viewed as the sum of many sine waves.

We saw last time that a sine wave which repeats with frequency f is generally of form,

$$P - P_{\text{atmos}} = \Delta P \sin(2\pi t f + \phi)$$

Here the 2π is needed because a “standard” sine wave repeats every 2π . The ϕ in there just tells whether the sine wave starts at zero right at time zero, or a little sooner or later. This wave repeats with period $T = 1/f$ because $\sin(2\pi T f) = \sin(2\pi)$ is the same as $\sin(0)$.

A sine wave with exactly twice the frequency of the first one, repeats every $T/2$. But that incidentally means that it also repeats every T ; where it is at a time t is the same as where it is at time $t + T$ because it repeats exactly twice in between. The same is true of a wave with 3 times the frequency, 4 times, and so on. This is illustrated in Figure 1.

If we add together one sine wave which repeats with period T , and another which repeats twice or three times as often, the resulting sum will repeat with period T . It will not repeat with period $T/2$ or $T/3$, since one of the waves which built it does not. However, all of them repeat with period T , so the sum will too.

To understand how to add together two waves, look at Figure 2, which illustrates the summation of two waves, one with 3 times the period of the other. You see how much the

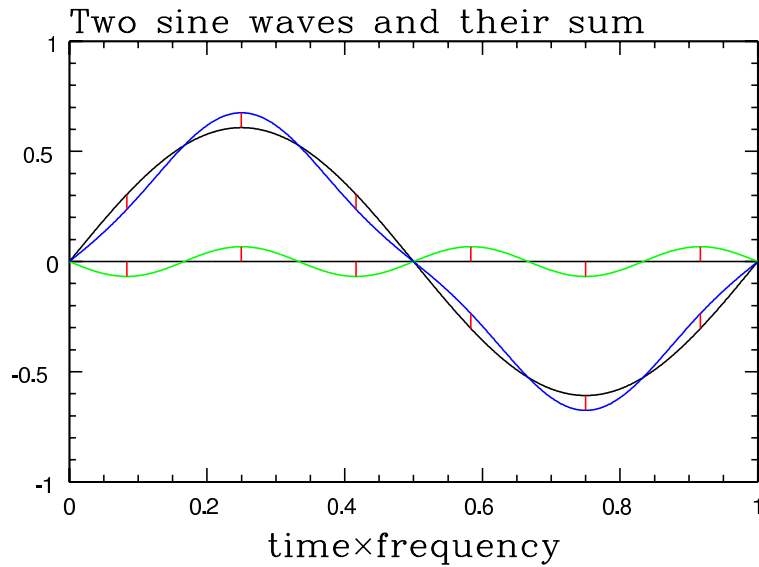


Figure 2: Addition of two sine waves, of period T and $T/3$, which form a wave of period T which is not a sine wave.

smaller wave moves up and down with respect to the axis, and you move up and down the same amount with respect to the larger wave.

Now for Fourier analysis:

- Any function of form $\sin(2\pi nft + \phi)$, with $n = 1, 2, 3, \dots$, has period $T = 1/f$.
- Any periodic function with period $T = 1/f$ can be expressed as a sum of such sine waves!

It is this last point which makes the subject useful. It's a pretty tall claim, so I should try to "prove it to you with examples."

First, let me explain the claim. Suppose you have the pressure pattern for an instrument or something, which is periodic with period $T = 1/f$. The claim is that there are coefficients P_1, P_2, P_3 , etc, and phases ϕ_1, ϕ_2, ϕ_3 etc, such that

$$\begin{aligned}
 P - P_{\text{atmospheric}} = & P_1 \sin(1 \times 2\pi ft + \phi_1) \\
 & + P_2 \sin(2 \times 2\pi ft + \phi_2) \\
 & + P_3 \sin(3 \times 2\pi ft + \phi_3) \\
 & + P_4 \sin(4 \times 2\pi ft + \phi_4) \\
 & + P_5 \sin(5 \times 2\pi ft + \phi_5) \\
 & + P_6 \sin(6 \times 2\pi ft + \phi_6) + \dots
 \end{aligned}$$

This certainly builds a function which is periodic with period T , since each sine wave repeats every T . That any curve can be built in this way is illustrated in Figure 3, which shows the construction of a triangle wave as the sum of sine waves.

The individual sine waves which build up the complex tone are called the **harmonics** of the tone. Everyone agrees that the sine wave with the same frequency as the tone is called the **fundamental** of the tone. There are two nomenclatures for the higher multiples. Either they are called:

- $2\times$ the frequency: 2'nd harmonic
- $3\times$ the frequency: 3'rd harmonic
- $4\times$ the frequency: 4'th harmonic

or sometimes,

- $2\times$ the frequency: 1'st harmonic
- $3\times$ the frequency: 2'nd harmonic
- $4\times$ the frequency: 3'rd harmonic

I think calling them the latter way is confusing, so in this class I will use the former—so the 7'th harmonic has 7 times the frequency of the fundamental. Be aware that some people will use the second nomenclature.

You will also hear people referring to the fundamental and the **overtones** (with the same confusion about nomenclature). For a periodic sound, “overtone” and “harmonic” are synonymous. The difference is that some non-periodic sounds (many percussion sounds, for instance) are composed as the sum of sine waves, but with the frequencies of the higher components not integer (counting number) multiples of the fundamental. In such case we call the other things overtones, and reserve the word “harmonic” for the case where they are integer (counting number, 1, 2, 3, 4, 5...) multiples of the fundamental.

A more nontrivial example is shown in Figure 4, which shows how successive Fourier components build even a very peculiar and complicated wave form, the sounds “ooo” or “eee” of the voice.

I should also show what it sounds like as you add one after another Fourier component to produce a complex wave. This is done in the sound files found in the HTML version of this lecture.

What happens when a complex tone arrives at your ear? The answer is that you can use linearity to think about what each harmonic does, separately. The behavior of the cochlea is *almost* linear. We will see later that it is not perfectly linear, but don't worry about that now. Therefore, for a complex tone, the excitation of the cochlea will be the excitations of the

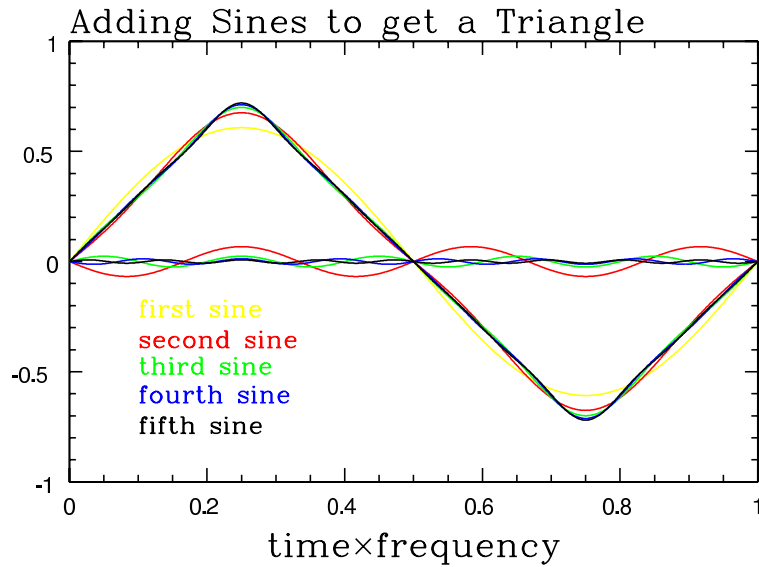


Figure 3: How a triangle wave can be built by adding up a series of sine waves. Each color shows a sine wave and the total wave after that sine has been added.

points corresponding to each harmonic, with sizes corresponding to how loud each harmonic is. This is illustrated in Figure 5. Your ear automatically recognizes this tower of spots of excitations as arising from a single complex tone. Your brain extracts the fundamental and gives this to you as “the” frequency, and you sense or feel the extra information about the relative loudnesses of the harmonics as the sound’s timbre.

Note that the harmonics get closer and closer together in terms of their pitch. This is because what is important to perception is the log of the ratio of frequency. The ratio of the 2’nd to 1’s harmonic is $2/1=2$; of 3’rd to 2’nd is $3/2=1.5$; of 4’t to 3’rd is $4/3=1.33$, and so on. [Musically, the separations are: $2/1$ =octave, $3/2$ =perfect fifth, $4/3$ =perfect fourth, $5/4$ =major 3’rd, $6/5$ =minor 3’rd, but the thirds are not tempered by more than the JND, as we will discuss in future.]

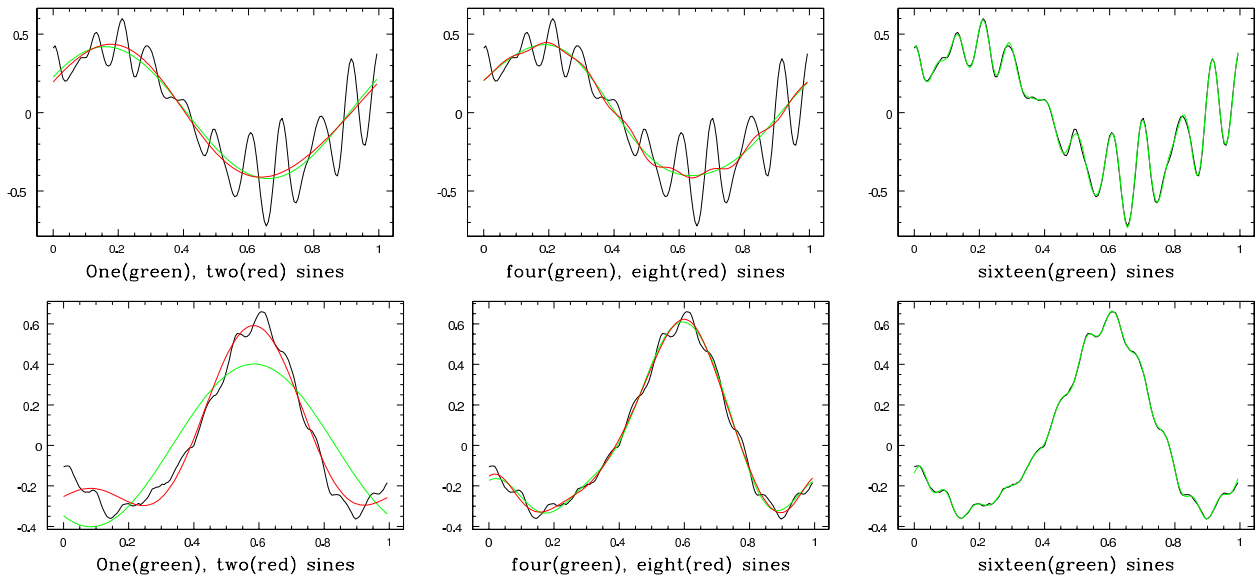


Figure 4: Top row: the sound “eee” fitted using different numbers of Fourier components. Bottom row: the same for “ooo.”

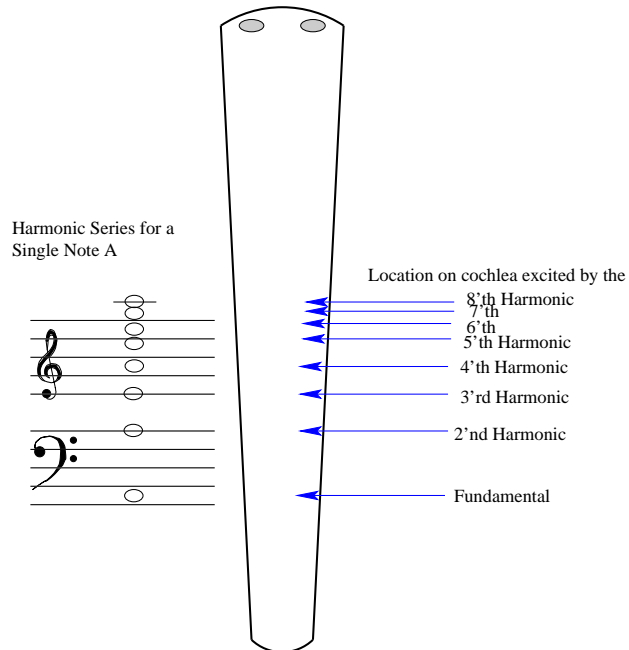


Figure 5: How the harmonic series of a complex note causes vibration in a series of points on your cochlea, at definite separations which your brain can interpret as a harmonic series.