Musical Acoustics Lecture Notes

Instructor: Guy D. Moore

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Preface

This document will collect some notes for topics discussed in Physics 225, "Musical Acoustics," where I consider the other references I have given you to be insufficient.

They will probably grow over the course of the term.

Contents

Chapter 1

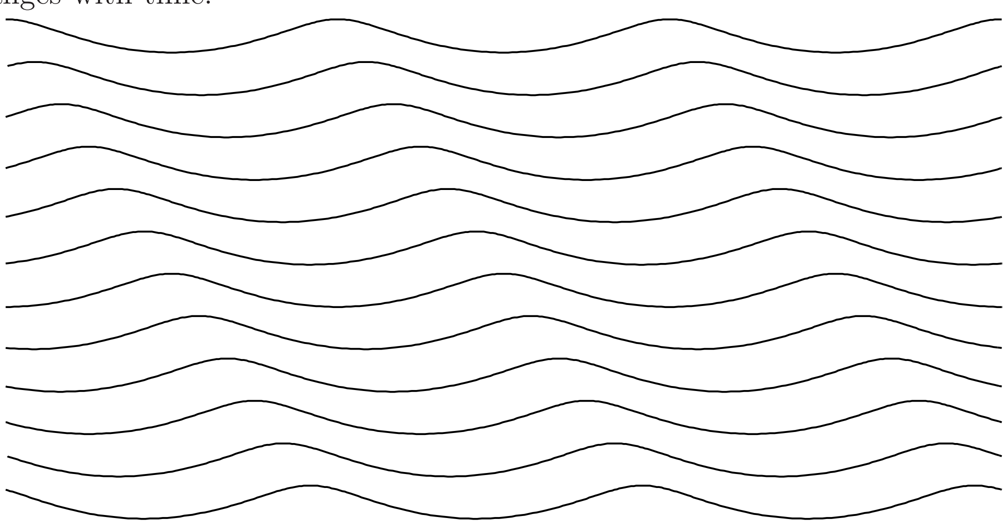
Waves in Air

A physicist's definition of a wave is:

> Wave: a pattern of disturbance in some medium which moves (propagates) even though the medium does not [much].

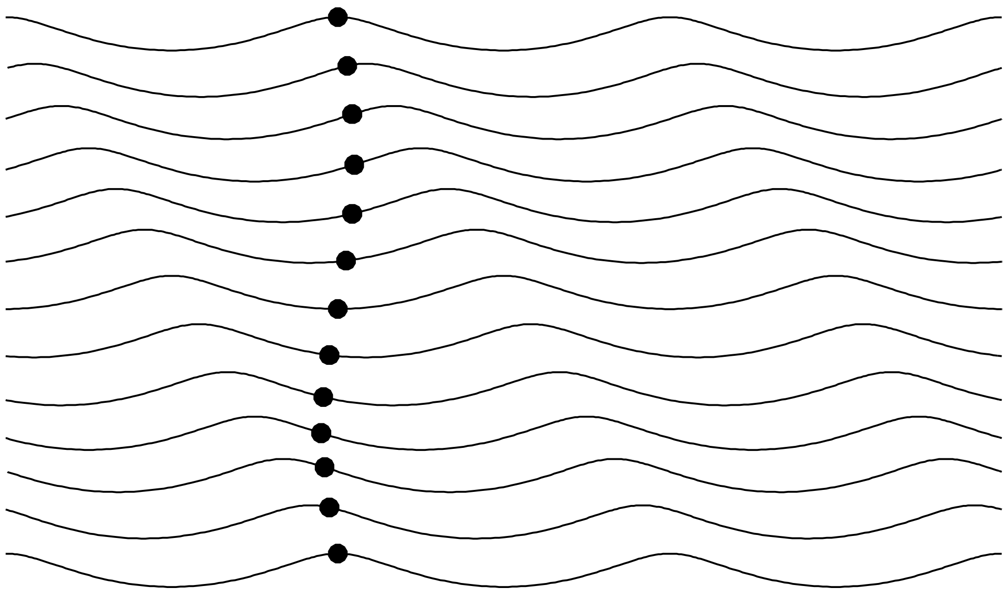
1.1 Water waves: a nice example

The easiest wave to visualize and understand is a (deep) water wave. The surface of the wave changes with time:

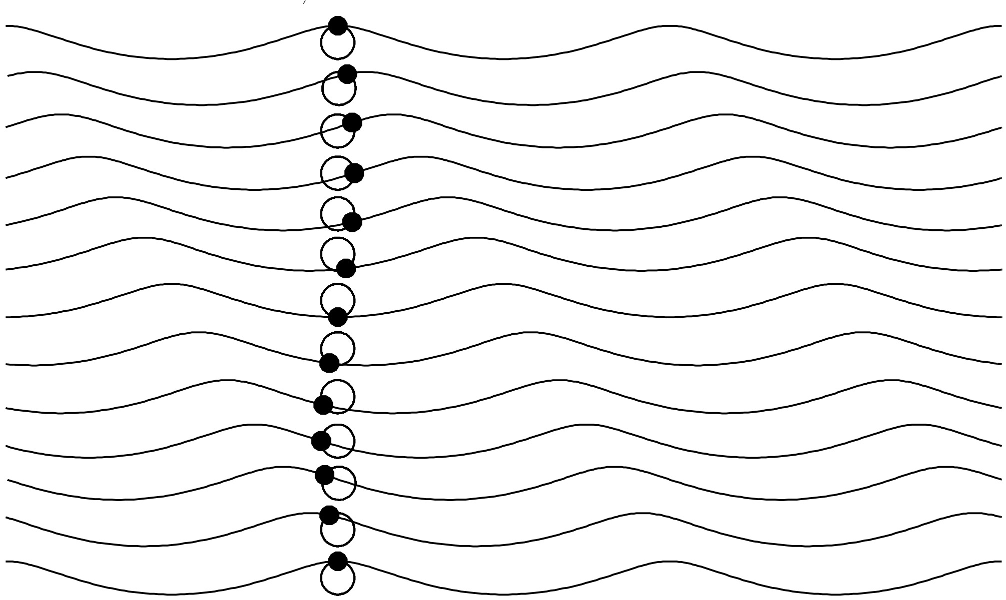


where I am showing a time series of what the wave looks like. Naively, we see that something is moving from left to right, by one "wavelength" (the distance between repetitions of the pattern, which for a sine wave is the distance between peaks) from top to bottom picture.

If however we put a cork on top of the water at some point, the time series instead looks like this:

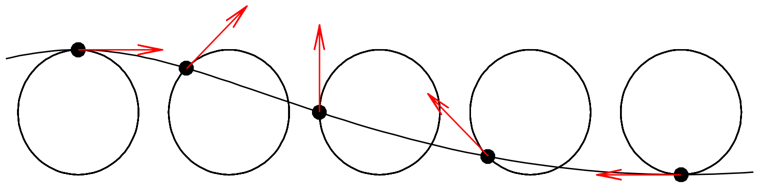


The cork traces out a circle, which is easier to see if I draw in the circle it is tracing out:



This example captures the central features of wave motion: the wave moves across the picture, but the water itself just goes around in little circles, so the cork ends up where it started. The wave consists of such circular motions for each little bit of water in the whole system, though of course different bits of water are at different points in their respective circular pattern.

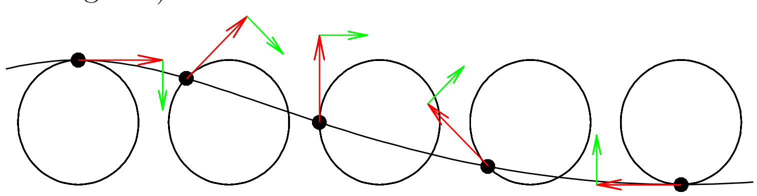
So let's redraw a snapshot of what is going on for one piece of the wave (a little more than half a wavelength), adding some more information: the height *and motion* of the water at several points along the wave.



That is, I have put 5 corks on the water and have shown the circles they trace out *and* the direction they are moving at any moment (which you can confirm by looking at the previous figure and seeing where the cork is going next when it is at each of the points shown). I did the arrows in red.

You can immediately see why the wave will move to the right. Following the arrows, each cork will soon be in the same position as the cork to its left is in right now. That means that the peak of the wave will move from the first to the second cork, and so forth.

The remaining question is why the water velocities should also change in the right way for the pattern to persist. For that to happen, each red arrow (velocity) has to be changing so that it will soon look like the red arrow of the cork to its left. That requires the following accelerations (shown in green):



Why should this happen? The first cork is at the top of the wave, and gravity is pulling it down. The last cork is at the bottom and buoyancy is pushing it up. The cork in the middle is on a "slope" so "of course" it will get pushed backwards (downhill). More accurately, it sees a higher pressure to its left than its right and so is pushed backwards.

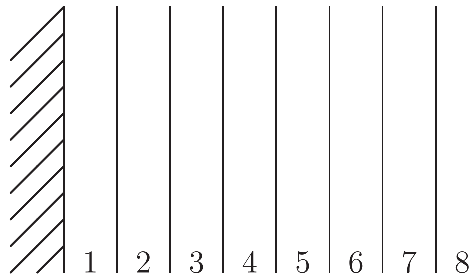
The motion of the water is what decides how the height will change. The height of the water is what determines which way the motion will change. The wave is a "self-moving" or "self-replicating" pattern where the motion shifts the pattern and the pattern shifts the motion.

1.2 Sound waves, qualitatively

Sound waves work the same way but using air. Ruthlessly stealing from the physics 224 notes:

Consider a chunk of air. At first, I will have a wall next to it, but then I will suddenly replace the wall with a high pressure region of air, pushing on it from its left. To understand what is happening, mentally divide the air into a series of layers. Remember that air molecules only travel a tiny distance between scatterings, so molecules generally stay within their layer.

The starting picture looks like this:



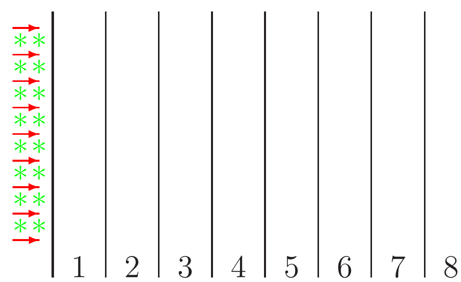
Recall that

1. Denser air has higher pressure, that is, it pushes harder.

2. When the forces on an object do not balance, the object accelerates.

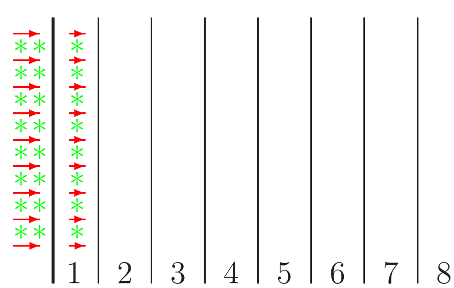
3. Once something is moving, it *keeps* moving until forces on it cause it to stop.

Now consider what happens when the wall is suddenly replaced by compressed, forwards moving air:

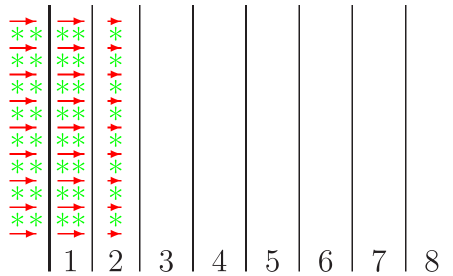


The green * are supposed to indicate that this air is compressed, that is, under higher pressure. The red arrows indicate what direction air is moving.

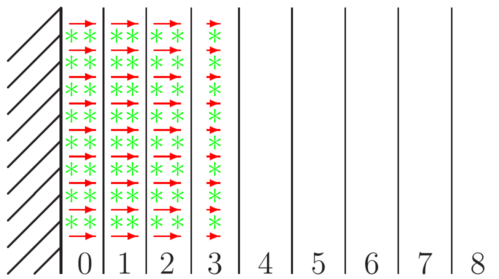
What happens to region 1? There is a larger pressure behind it than in front of it. Therefore the forces do not balance, and so it will start to move forward. Since the air behind it is moving forward, it will also get squeezed into a smaller space. Therefore, a moment later, the situation will look like this:



The first layer of air has become compressed, and has started to move forward. Since it is now compressed, it pushes harder on the second layer, than it did before. Since it is moving forward, it compresses the second layer. Therefore, a moment later the situation will look like,

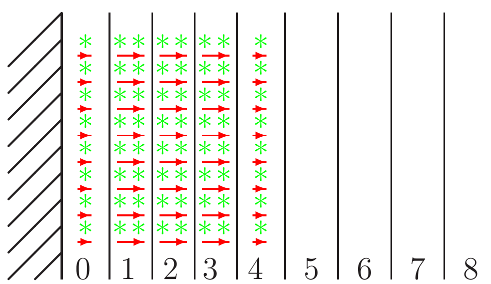


Now suppose a moment later that the barrier re-appears. Since the air moved forward, I will put it behind the new layer that got brought in. The situation looks like this:

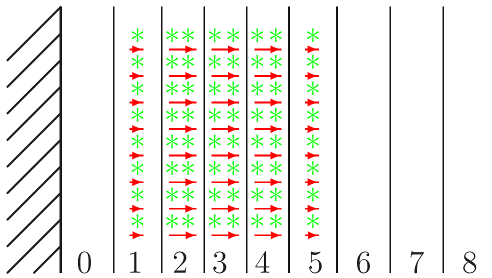


Notice that the air has only moved forward a bit–but the region where the air is compressed, is pretty big now.

Next what? The air in region 1 is feeling extra pressure in front but also behind. The pressures balance on it. But once something is moving forward, it keeps moving forward. Similarly, regions 0 and 2 keep moving forward, and regions 3 and 4 are being pushed forward and move faster. Since the wall is back in place, region 0 will now be getting stretched out, which means its pressure will fall. So we will get,



Now regions 4 and 5 will get compressed and pushed forward, but regions 0 and 1 feel more pressure on their fronts than their backs, and will keep moving forward (and hence, will de-compress) and will be slowed down. Therefore we will get,



From now on, the compressed region has nothing to do with the presence of a wall behind it. The front of the region sees a higher pressure behind than in front; so it speeds up. The back region sees a higher pressure behind than in front; so it slows down. Since the middle is moving, the front gets compressed and the back gets de-compressed. That is just the right thing to keep moving the pattern forward.

1.3 Sound waves, quantitatively

Now let us do the sound waves quantitatively.

The simplest problem to work with is a sound wave where the air pressure and velocity depend only on time and one direction, say the x direction. Also consider variation with is sinusoidal. Then the general appearance of a sound wave will be

$$v_x = v_0 \sin\left(2\pi\left[\frac{x}{\lambda} - ft\right]\right) , \tag{1.1}$$

$$P = P_{\rm atmos} + P_0 \sin\left(2\pi\left[\frac{x}{\lambda} - ft\right]\right) . \tag{1.2}$$

This is a forward moving wave. The wave length is λ and the frequency is f, meaning that in a distance $x = \lambda$ the wave repeats itself (since $\sin(2\pi)$ repeats $\sin(0)$) and in a time of $t = 1/f$ the wave also repeats.

The sound speed is given by $c_{\rm s} = \lambda f$. We can "build" a wave which has whatever value of pressure fluctuation P_0 and whatever wavelength λ we choose, but *dynamics* will then determine what the right values of v_0 the peak velocity and f the frequency will be. That is, we need to do some physics to figure out these two unknowns.

The first piece of physics is Newton's second law,

$$F = ma \tag{1.3}$$

Acceleration means the rate of change of velocity. I will use calculus even though it is not a prerequisite; if you want a detailed explanation please see me in person (all but 1 student said they had seen calculus). If we think about a little volume of air, of length L in the x direction and area A in the $y - z$ plane, the mass inside is

$$m = \rho V = \rho A L$$

where ρ is the density of the air. The density equals $\rho = n m_{\rm molec.}$ with n the density of molecules (number per cubic meter) and $m_{\rm molec}$ the (average) mass of a molecule. For now take ρ to be a known constant.

Now we need to know the force. Force due to pressure equals pressure times area. Therefore the force is

$$F = AP_{\rm behind} - AP_{\rm in\ front} = -AL\frac{P_{\rm in\ front} - P_{\rm behind}}{L} \qquad (1.4)$$

The combination AL is the volume V. In the limit of small L, the ratio becomes the x derivative of the pressure:

$$F = -V\frac{dP}{dx} = V\rho\frac{dv}{dt} \qquad \text{or} \qquad \boxed{\rho\frac{dv}{dt} = -\frac{dP}{dx}}. \qquad (1.5)$$

(Careful, capital V is volume, little v is velocity of the air.) [Comments in brackets are for people with more advanced backgrounds. In general in 3 dimensions this should instead read $-\vec{\nabla} P = \rho d\vec{v}/dt$, it is a vector expression.]

Performing these derivatives on our "guesses" for the form of v and P, we obtain

$$\begin{aligned} \rho\frac{dv}{dt} &= -\frac{dP}{dx} \\ 2\pi\rho v_0 f \cos\left(2\pi\left[\frac{x}{\lambda} - ft\right]\right) &= -\frac{2\pi P_0}{\lambda}\cos\left(2\pi\left[\frac{x}{\lambda} - ft\right]\right), \\ \frac{P_0}{v_0} &= \rho\lambda f = \rho c_{\rm s}\,. \end{aligned} \qquad (1.6)$$

That means that (if only we knew $c_{\rm s}$) we would know the relation between pressure and velocity in a sound wave. It does not depend on the wavelength; only on two properties of the medium, the density and the speed of sound.

The ratio of pressure to velocity is important. The combination of physical properties which determines it, $\rho c_{\rm s}$, is important and will play a central role in what is to come, and so we will give it a name. It is called $P_0/v_0 = \rho c_{\rm s} =$ **Mechanical Impedance**. In general, impedance is a ratio of "how hard you push on something" to "how much it moves." It gets its name because intuitively, it is a mechanical property of the substance (in this case, the air) which tells how much it resists ("impedes") motion. It has analogues in many many systems (all systems which display wave phenomena).

The other piece of physics we need is the one which describes how pressure changes as a gas is compressed. For air this is very accurately described by the ideal gas law. For chemists this reads PV=NRT, with P the pressure, V volume, N the number of moles of gas, R some constant, and T the temperature in Kelvin. For physicists, the "right" way to measure temperature is as an energy (Joules). You can write things in terms of Kelvins if you keep the conversion factor between Kelvins and Joules:

$$1 \text{ Kelvin} = 1.38 \times 10^{-23} \text{ Joule}\,. \qquad (1.7)$$

Defining $k_{\mathrm{B}} = 1.38\times 10^{-23}$J/Kelvin, you can write a temperature in Kelvins as a temperature; $k_{\mathrm{B}}T$ is the temperature expressed in Joules.

Using this definition and writing in terms of the density of atoms, this gas law is

$$P = nk_{\mathrm{B}}T\,. \tag{1.8}$$

The pressure is the product of the number of molecules and the temperature expressed as an energy (the temperature in Kelvins times the conversion from Kelvins to Joules, k_{B}).

There is an easy way to understand why. Consider one molecule in a box of length L and area A. It bounces back and forth between the walls. Each time it hits the right wall, it imparts a momentum $2mv_x$. (The 2 is because it reverses direction.) The frequency with which it does this is $v_x/2L$ (since the time to go across the box and back is $2L/v_x$). Therefore the average force it exerts is mv_x^2/L, and the average pressure is $mv_x^2/(LA) = mv_x^2/V$. If we have some density n of such molecules, the pressure they exert will be $Nmv_x^2/V = nmv_x^2$ (N the total number of molecules, $N/V = n$ the density). Now $\frac{1}{2}mv_x^2$ happens to be the amount of *energy* the molecule carries in x motion, and this is related to the temperature (which is an amount of ENERGY) by $mv_x^2/2 = T/2$ or $mv_x^2 = T$. That is, the meaning of temperature is the average amount of energy stored in each kind of motion which can carry energy (times 2). Therefore the pressure will be nT if T is expressed in energy, or $nk_{\mathrm{B}}T$ if it is expressed in Kelvins.

The air's motion compresses or rarefies the air, changing the number density and therefore the pressure. Consider a chunk of air of length L and area A, hence volume V. The air on either side is moving; if it is moving "in" on the region, it will get compressed. How fast is that volume changing, given that the "sides" of the chunk are moving?

$$\frac{dV}{dt} = \frac{d(AL)}{dt} = A\frac{dL}{dt} \tag{1.9}$$

where I used that things are only moving in the x direction. How fast is the length changing? That is set by the velocity at the start and end of our "chunk" of air:

$$\frac{dV}{dt} = A\frac{dL}{dt} = A(v_{\mathrm{front}} - v_{\mathrm{behind}}) = AL\frac{v_{\mathrm{front}} - v_{\mathrm{behind}}}{L} = V\frac{dv}{dx}\,. \tag{1.10}$$

Here I used the same trick that I used with the pressure to identify the space dependence of velocity. [In general dv/dx should be $\vec{\nabla}\cdot\vec{v}$.] The useful way to write this is

$$\frac{1}{V}\frac{dV}{dt} = \frac{dv}{dt}\,. \tag{1.11}$$

Now n is proportional to $1/V$. Elementary calculus shows that

$$\frac{1}{n}\frac{dn}{dt} = -\frac{dv}{dt}\,. \tag{1.12}$$

If the volume increases 1%, then the number density gets diluted 1%, and if the volume is reduced 1% the number density increases 1%.

So how does the pressure change?

$$\frac{dP}{dt} = \frac{d(nk_{\rm B}T)}{dt} \,. \tag{1.13}$$

$k_{\rm B}$ is a constant and can be pulled out. You might think that the temperature does not change, in which case $dP/dt = k_{\rm B}T dn/dt = -nk_{\rm B}T dv/dx = -P dv/dx$. But the temperature *does* change. Compressing gas heats it up. If we compress gas of pressure P by a fractional volume $\Delta V = A\Delta L$, that requires *WORK* of (force times distance is) $F\Delta L = PA\delta L = P\delta V$. Putting in that work will add energy to the gas, which will heat it up. How much will it heat the gas? We already saw that the gas stores energy in $mv_x^2/2 = T/2$ for each molecule. The same is true of v_y^2 and v_z^2, so the energy in the motion of molecules is $\frac{3}{2}T$ per molecule. On top of that, for molecular (as opposed to atomic) gases, there are usually rotational and possibly vibrational modes which can be excited. For the diatomic molecules which make up air, there are also 2 ways they can rotate which each take $T/2$ of energy per molecule, so the energy per molecule is $\frac{5}{2}T$. (At very high energies vibrational motions also come into play. But we only care about air at room temperature.)

The bottom line is that if you compress a gas 1%, you add $T/100$ of energy per molecule in the gas; but since the gas energy is $5T/2$ per molecule, this only changes the temperature by 2/5 of 1%. The result is that the effect of the temperature rise is 2/5 as large as the effect of the density change, and

$$\frac{dP}{dt} = k_{\rm B}\frac{d(nT)}{dt} = -k_{\rm B}\left(1+\frac{2}{5}\right)nT\frac{dv}{dx} = -\frac{7}{5}P\frac{dv}{dx} \,. \tag{1.14}$$

The ratio 7/5 [which, for chemists, is c_p/c_v the ratio of constant pressure to constant volume heat capacities] is called γ:

$$\boxed{\frac{dP}{dt} = -\gamma P\frac{dv}{dx}} \,. \tag{1.15}$$

Applying this relation to our guess for how v and P should behave, we obtain

$$\begin{aligned} \frac{dP}{dt} &= -\gamma P_{\rm atmos}\frac{dv}{dx} \,, \\ -2\pi f P_0 \cos\left(2\pi\left[\frac{x}{\lambda} - ft\right]\right) &= -2\pi\gamma P_{\rm atmos}\frac{v_0}{\lambda}\cos\left(2\pi\left[\frac{x}{\lambda} - ft\right]\right) \\ \frac{P_0}{v_0} &= \frac{\gamma P}{\lambda f} = \frac{\gamma P_{\rm atmos}}{c_{\rm s}} \,. \end{aligned} \tag{1.16}$$

We already had another relation for this ratio:

$$\frac{P_0}{v_0} = \rho c_{\rm s} = \frac{\gamma P_{\rm atmos}}{c_{\rm s}} \quad \Rightarrow \quad c_{\rm s}^2 = \frac{\gamma P_{\rm atmos}}{\rho} \,. \tag{1.17}$$

Therefore we learn the speed of sound, which also turns out to be a property simply of the nature of air and not of the specifics of the wave.

Note that the ratio $P_{\rm atmos}/\rho$ simplifies if we also apply the ideal gas law: $P = nk_{\rm B}T$ so $P/\rho = (n/rho)k_{\rm B}T$. But $\rho = nm_{\rm molec}$. So the speed of sound is $c_{\rm s} = \sqrt{\gamma k_{\rm B}T/m_{\rm molec}}$. It depends on the temperature of the gas, and on what kind of molecules make it up (that determines γ and $m_{\rm molec}$), but it does not depend on the density of the gas; that turns out to cancel out.

Now let us plug in numbers. At room temperature $T = 293$ Kelvin, we saw $k_{\rm B} = 1.38065 \times 10^{-23}$J/Kelvin, a Joule is kg m^2/s^2. The mix of 78% Nitrogen, 21% Oxygen, and 1% Argon[1] which we breathe has on average $m_{\rm molec} = 29$ Atomic Mass Units. An Atomic Mass Unit is 1.6605×10^{-27} kg. Air has $\gamma \simeq 7/5$. Therefore the speed of sound is $c_{\rm s} \simeq 344$ m/s (give or take). The speed is higher in hot air and (to a lesser extent) in moist air (since $m_{\rm molec} = 18$ for water molecules).

Let us rewrite our key equations:

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{1}{\rho}\frac{dP}{dx}\,, \\
\frac{dP}{dt} &= -\gamma P_{\rm atmos}\frac{dv}{dx}\,. \qquad (1.18)
\end{aligned}$$

If we were being really careful (or picky), we would say that $P_{\rm atmos}$ should really be the *total* pressure, not just the average value; that is, it should include the varying pressure from the wave itself. Similarly, the density should include the density variation of the gas. Also in deriving the first equation I sneakily made a small velocity expansion by not worrying about how the gas is moving through the little volume element we thought about when we computed forces. The upshot is that if v_0 and P_0 are not tiny compared to $c_{\rm s}$ and $P_{\rm atmos}$, then there are important corrections. These corrections are *nonlinear*: they mean that the equations describing v and P do not just involve these variables at first (linear) order, but products of these things.

Nonlinear equations are nasty. Sound waves scatter from each other; the presence of one sound wave changes the way another sound wave behaves. But for "quiet" sound waves, we see that the equations *are* linear, which means that we can figure out what each sound wave will do by itself without having to worry about the effects of any other sound waves running around. This is a major simplification which makes the consideration of room acoustics possible.

[1]This is dry air. Water vapor varies from 0 to a few percent. Carbon dioxide and other gases are negligible.

Chapter 2

Intensity and power

One of the most important properties of waves is that they carry energy from place to place. For sound waves, this is how we can hear them (the energy they deliver to our ears) and it is a key aspect of sound perception. It is also essential in understanding how musical instruments work. Therefore we will take some time to understand it properly.

2.1 Work, power, and intensity

First, recall some essential definitions:

Energy: The stuff that makes things move. The energy of a moving object is $\frac{1}{2}mv^2$. Energy can also be stored in heat, in gravitational potential (pushing something uphill), in elastic strain (as in a spring), and so forth.

Work is an amount of energy expended in doing something. The work involved in applying a force to a moving object is force times distance, $\vec{F} \cdot \Delta\vec{x}$.

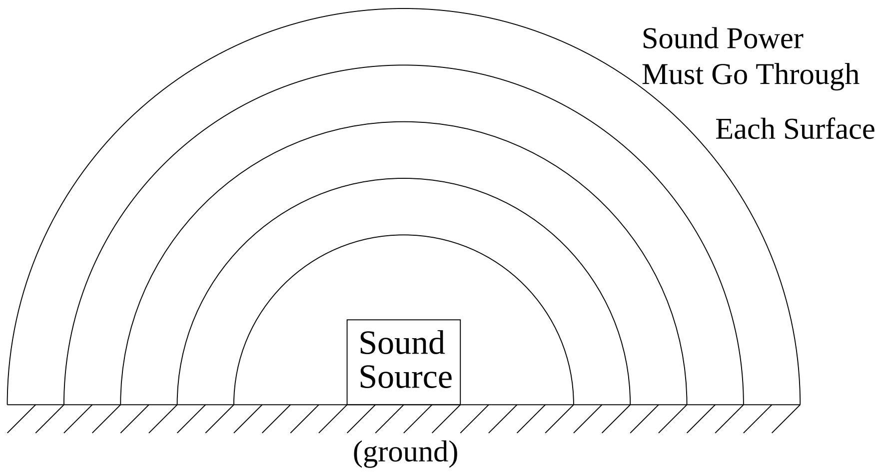
Units: Energy is measured in Joules, with $1\text{J} \equiv 1\frac{\text{kg m}^2}{\text{s}^2}$. You can read off these units from the expression $E = \frac{1}{2}mv^2$.

Power: The rate at which energy is delivered. Equivalently, power is an amount of energy per unit time, Power= dE/dt. The power consumed in applying a force is $\vec{F} \cdot \vec{v}$ with v the velocity of the thing one is applying a force to. The energy delivered by a steady power in an amount of time Δt is $E = \text{Power} \times \Delta t$.

Units: Power is measured in Watts, $1\text{W} = 1\text{J/s} = 1\frac{\text{kg m}^2}{\text{s}^3}$.

These should both be familiar to you from some physics class.

The new quantity we need is intensity. A sound source consumes energy, and we can talk about how fast it is releasing sound energy as a power. That is, sound sources produce some amount of sound power. [Amplifier/speaker systems have power ratings, which represent the maximum amount of electrical power they can draw. This is not the same as the maximum sound power they can produce, which is found by multiplying by an efficiency, which is usually much less than 1.] However, the sound goes out to fill space. Therefore it is more convenient to talk, not of the energy in the sound wave, but of the energy density in a sound wave, which is energy per unit volume (an intensive quantity). Its units are $\mathrm{J/m^3 = kg/m\,s^2}$. Similarly, since the sound power is conveyed by the sound wave across a surface, it is best to think about it in terms of the power per unit area which is crossing a surface. To illustrate:



The power per area is called

Intensity: Power per unit area crossing some surface. This is an intensive quantity which describes how loud a sound wave is at some point in space. Intensity is (almost) what you ear, or a microphone, measures to determine the loudness of a sound.

Units: Power/area is $\mathrm{W/m^2 = kg/s^3}$. These units look bizarre, but that is life. There is no special name for $\mathrm{W/m^2}$.

2.2 Intensity of a sound wave

Since intensity has a special role in sound perception and measurement, we should figure out how it is related to the physical properties of a wave.

The quick and dirty way is to note that, since power is force times velocity, intensity

$$I = \frac{\vec{F}\cdot\vec{v}}{A} = \frac{\vec{F}}{A}\cdot\vec{v} = Pv \tag{2.1}$$

where A is area, P is pressure and v is (air) velocity. You might worry that in the last expression I turned vectors into scalars, by recognizing that the force per area in air is just the pressure. This step is fair provided that you are talking about a single sound wave going in one direction.

So consider the sound wave from the last chapter, Eq. (1.1) and Eq. (1.2). The intensity will be

$$I = Pv = (P_{\rm atmos} v_0) \sin\left(2\pi \left[\frac{x}{\lambda} - ft\right]\right) + P_0 v_0 \sin^2\left(2\pi \left[\frac{x}{\lambda} - ft\right]\right) . \qquad (2.2)$$

The first term averages to zero over time. It is also nonsense. To understand why I say that, imagine that the air were moving, $v_x = v_{\rm wind} + v_0 \sin(\ldots)$. Then we would have also found a term $v_{\rm wind} P_{\rm atmos}$, and it does *not* average to zero. But it just reflects the fact that the air itself is moving, and the air carries some internal energy density given by P. So this term, and the $P_{\rm atmos} v_0$ term, just reflect the fact that this internal energy of the air is getting moved around by the motion of the air. This is *not* the transport of energy by sound itself; sound is the phenomenon where energy gets transported even though the air ends up where it started. Therefore the intensity *because of sound* is

$$I = P_0 v_0 \sin^2\left(2\pi \left[\frac{x}{\lambda} - ft\right]\right) , \qquad \text{or in general,} \qquad I = v\Delta P \qquad (2.3)$$

where ΔP means the *part* of the pressure due to the sound wave, $P = P_{\rm atmos} + \Delta P$ (if you prefer, it is the pressure subtracting off the average "baseline" pressure).

Now we use our result $\Delta P / v = \rho c_{\rm s}$ to rewrite this two other ways:

$$I = \rho c_{\rm s}\, v^2 \, , \qquad (2.4)$$

$$I = \frac{1}{\rho c_{\rm s}} (\Delta P)^2 \, . \qquad (2.5)$$

The quantities above are the *instantaneous* intensities, that is, the intensity of sound at a point in space at an instant in time. Generally we really want an average. For instance, your ear does not tell your brain what the intensity of sound is once every millisecond; it averages the intensity of sound over some time scale which is more like 30 milliseconds. (This is like the fact that movies flicker, but you see a smooth image because your eyes average the light brightness over about 50 milliseconds.) Now 30 milliseconds sounds short, but it is longer than the period of almost any musical (or audible) sound. Therefore the intensities we usually care about are averaged over the period of the oscillations in the sound wave. Therefore, we should really say

$$I = \rho c_{\rm s} \, \langle v^2 \rangle \, , \qquad (2.6)$$

$$I = \frac{1}{\rho c_{\rm s}} \langle (\Delta P)^2 \rangle \, . \qquad (2.7)$$

Here $\langle \ldots \rangle$ means "average of ... over time." (We usually do *NOT* want the average of intensity over space. This is because your ear, and most microphones, are small, smaller than the wave lengths of the sounds you usually deal with, which range from centimeters to meters.)

Note that $\langle \sin^2(\ldots) \rangle = \frac{1}{2}$. The easy way to understand that is that $\sin^2(\theta) + \cos^2(\theta) = 1$. Averaging cannot change this: $\langle \sin^2 \rangle + \langle \cos^2 \rangle = 1$. But cosine is just sine, shifted by $\pi/2$

phase. So they must have the same average square. Therefore the average square of each must be 1/2. This gives *For Sine Waves*

$$I = \frac{1}{2} P_0 v_0 = \frac{\rho c_s}{2} v_0^2 = \frac{1}{2\rho c_s} P_0^2 \,, \qquad \text{For sine waves} \,. \tag{2.8}$$

Also for sine waves, the distance that the air moves, back and forth, is

$$v = v_0 \sin \left(2\pi \left[\frac{x}{\lambda} - ft \right] \right) \quad \Rightarrow \quad \Delta x = \frac{v_0}{2\pi f} \cos \left(2\pi \left[\frac{x}{\lambda} - ft \right] \right) \,. \tag{2.9}$$

To check this, take the time derivative of the air displacement Δx, which should be the air velocity v. You can use this to find out how far back and forth the air moves.

RMS velocity: The *root mean squared* or RMS velocity is the square root of the average of the square: $v_{\mathrm{RMS}} \equiv \sqrt{\langle v^2 \rangle}$. Same with the RMS anything else; $\Delta P_{\mathrm{RMS}} \equiv \sqrt{\langle \Delta P^2 \rangle}$.

The RMS value is a convenient measure of the "typical" air pressure or velocity fluctuations.

2.3 Energetics of a wave

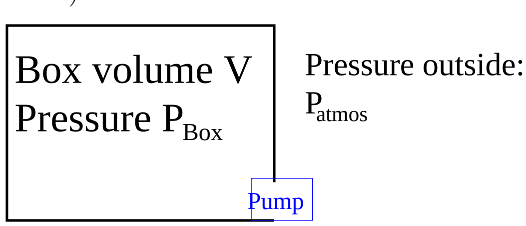
Let us try to understand Eq. (2.6) and Eq. (2.7) by thinking about the energy in the sound wave.

The presence of a sound wave means that the air is moving and has pressure fluctuations. Both things cost energy, so there will be an *energy density* associated with the wave (an energy per unit volume reflecting the extra energy in the air because of the presence of the sound wave). It is easy to figure out the *part* of this energy density because of the air motion. Since kinetic energy (energy of motion) is $E = mv^2/2$, the energy in a small volume V of air is $E = (\rho V)m^2/2$, since the mass is $m = \rho V$. This contributes an energy density of

$$\frac{E}{V} = \frac{\rho}{2} v^2 + \text{Pressure piece.} \tag{2.10}$$

Now we just need to figure out how much energy is stored because of the compression and rarefaction (decompression) of the air.

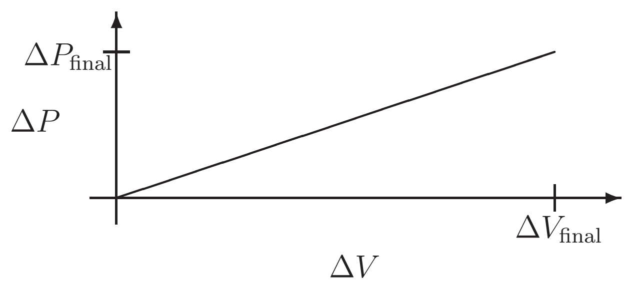
To figure out what the energy cost of compressing air is, consider a box of volume V, initially filled with air at normal pressure. A small pump can push outside (standard pressure) air into the box:



The pump moves air from the outside, where $P = P_{\mathrm{atmos}}$, to the inside, there $P = P_{\mathrm{box}}$.

Each bit δV of air the pump moves requires a work equal to $(P_{\rm box} - P_{\rm atmos})\delta V$. (To convince yourself, imagine the bit of air is a tiny box of area A and length L. The pump moves it a distance L (so it goes from outside to inside the box): the force the pump must apply is difference between the pressure forces pushing it out and pushing it in, $AP_{\rm box} - AP_{\rm atmos}$. The work is force times distance, $(AL)(P_{\rm box} - P_{\rm atmos})$. But AL is the volume δV.)

For the first bit of air, the pressure in the box and the pressure outside are equal. As each bit of air is shoved into the box, the pressure inside rises; so each piece takes more work to get it into the box. Name the total amount of air moved is $\Delta V_{\rm final}$. Name the final pressure difference $\Delta P_{\rm final}$. The way ΔP changes with ΔV is



The area under the curve is the total work done. This is *half* of $\Delta P_{\rm final}\Delta V_{\rm final}$. The last bit of air works against the full pressure difference ΔP, but the first bit of air works against no pressure difference; averaging, the air works against half of the final pressure difference. So the work is $\frac{1}{2}\Delta P_{\rm final}\Delta V_{\rm final}$.

It remains to relate $\Delta P_{\rm final}$ to $\Delta V_{\rm final}$. You would think that a 1% increase in the amount of air in the cavity ($\Delta V = V/100$) would lead to a 1% increase in the pressure ($\Delta P = P_{\rm atmos}/100$). But this is wrong, because the air heats up as it is compressed; there is an extra factor of γ (same as last chapter):

$$\Delta P = \gamma P_{\rm atmos} \frac{\Delta V}{V} \, . \tag{2.11}$$

Therefore the work is

$$\text{Work} = \frac{1}{2}\Delta P_{\rm final}\Delta V_{\rm final} = \frac{\gamma P_{\rm atmos}}{2} \frac{(\Delta V_{\rm final})^2}{V} = \frac{1}{2\gamma P_{\rm atmos}} V (\Delta P_{\rm final})^2 \, . \tag{2.12}$$

The last expression is most convenient. It means that the energy density in compressed air is

$$\text{Compression contrib. to} \quad \frac{E}{V} = \frac{1}{2\gamma P_{\rm atmos}} (\Delta P)^2 \, . \tag{2.13}$$

The total energy density stored in a wave is therefore

$$\frac{E}{V} = \frac{\rho}{2} v^2 + \frac{1}{2\gamma P_{\rm atmos}} (\Delta P)^2 \, . \tag{2.14}$$

Now it happens, as we have seen, that $\Delta P = \rho c_{\rm s} v$. Therefore the second term equals

$$\frac{1}{2\gamma P_{\rm atmos}}(\Delta P)^2 = \frac{1}{2\gamma P_{\rm atmos}}(\rho^2 c_{\rm s}^2 v^2) = \frac{\rho}{2} c_{\rm s}^2 \frac{\rho}{\gamma P_{\rm atmos}} v^2 = \frac{\rho}{2} v^2 \, . \tag{2.15}$$

In the next to last step I grouped terms in a suggestive way; in the last step I used that $c_{\rm s}^2 = \gamma P_{\rm atmos}/\rho$ so $c_{\rm s}^2$ cancels the other term. The punchline is that

> In a sound wave, the energy stored as air motion and the energy stored in compression of the gas are equal.

That means that the total energy density is $\frac{E}{V} = \rho v^2$.

What does it all have to do with intensity? Consider a tube of air, area A and length L. The energy stored inside is $E = V \rho v^2 = AL\rho v^2$. The time it takes for all of this energy to move the length of the tube is $t = L/c_{\rm s}$. That means, in this amount of time, the energy all moves out through the end of the tube. Therefore thepower of energy moving out through the end of the tube is

$$\text{Power} = \frac{E}{t} = \frac{AL\rho v^2}{L/c_{\rm s}} = A\rho c_{\rm s} v^2 \tag{2.16}$$

and the intensity is

$$I = \frac{\text{Power}}{A} = \rho c_{\rm s} v^2 \, . \tag{2.17}$$

So we see that the expression we got earlier is just telling about how fast the energy of a sound wave moves through the air. We would get the expression involving $(\Delta P)^2$ if we expressed the energy density as $\frac{E}{V} = \frac{1}{\gamma P_{\rm atmos}}(\Delta P)^2$ and played the same game.

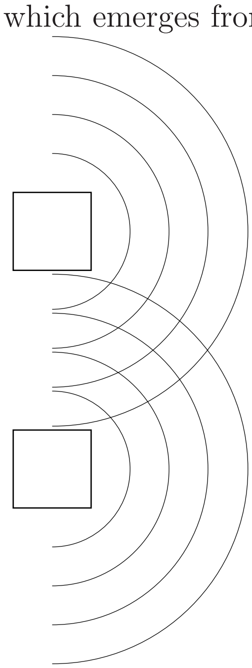
Chapter 3

Adding two waves

First consider the problem where there are two sound sources which are producing coherently. By “coherently” I mean that the sound wave each one produces is in some sense the same as the sound wave the other produces; for instance, they are two speakers driven by the same amplifier, or they are really one sound source and an image of that sound source due to some sound-reflecting surface such as a wall.

For simplicity consider the case where the sound source produces a sine wave. I say “for simplicity” but in fact this is absolutely general; any real sound can be broken down into the sine waves which make it up (see Phys. 224 notes Chapter 8) and we can figure out what happens to each sine wave separately. Also, your ear separates the sounds out into these sine waves, so it is actually the right approach for understanding what you will hear, too.

Start out by considering two sources: for fun assume that they are speakers being driven by the same amplifier, so the sound which emerges from each one is the same:



Suppose one speaker pushes the air back and forth at frequency f. Assume that at time $t = 0$ the "phase" is ϕ, which just means that the pressure is zero and heading negative right at the time $t = \phi/(2\pi f)$. If I am a distance r from one speaker, I hear the pressure rise and fall according to

$$P_{\rm one\ source} = \frac{A}{r} \sin\left(2\pi\frac{r}{\lambda} - 2\pi f t + \phi\right) , \tag{3.1}$$

where A is some number determined by the loudness of the source; the factor A/r correctly includes the way the sound wave dies away with distance such that $I \propto P^2_{\rm one\ source} \propto 1/r^2$. [The symbol $\propto$ means "is proportional to."] Here f and λ are the frequency and wavelength of the sound, related by $f\lambda = c_{\rm s}$ the speed of sound.

I should clarify the things inside the argument of the sine, above. The factor $-2\pi f t$ is there to reflect the way that the pressure produced by the source rises and falls with time. The factor $2\pi r/\lambda$ is there because I am at some distance from the speaker. It takes time for the sound to get from the speaker to me; so I am really hearing the sound the speaker produced some time ago. The time it takes for the sound to get from the speaker to me is $t_{\rm travel} = r/c_{\rm s}$ (since sound travels at the speed of sound). Therefore I am hearing the sine wave which the speaker produced, not at the current time t, but at the past time $t - t_{\rm travel}$. So another way to write the sine wave above would be

$$\sin\left(-2\pi f(t - t_{\rm travel}) + \phi\right) = \sin\left(-2\pi f t + 2\pi f t_{\rm travel} + \phi\right) = \sin\left(-2\pi f t + 2\pi f\frac{r}{c_{\rm s}} + \phi\right) . \tag{3.2}$$

Now because $f\lambda = c_{\rm s}$, then $f/c_{\rm s} = 1/\lambda$ and I can rewrite this as

$$\sin\left(-2\pi f(t - t_{\rm travel}) + \phi\right) = \sin\left(2\pi\frac{r}{\lambda} - 2\pi f t + \phi\right) . \tag{3.3}$$

In other words, the r/λ term is just keeping track of the fact that, at some distance from a source, the sound you receive is the sound the source produced at some time in the past.

Assume to make life easier that the two speakers have the same amplitude A and the same phase ϕ (the usual case if they are driven by the same amplifier). In general a listener might be at distance r_1 from one speaker and r_2 from the second speaker. Then the pressure the listener would hear would be

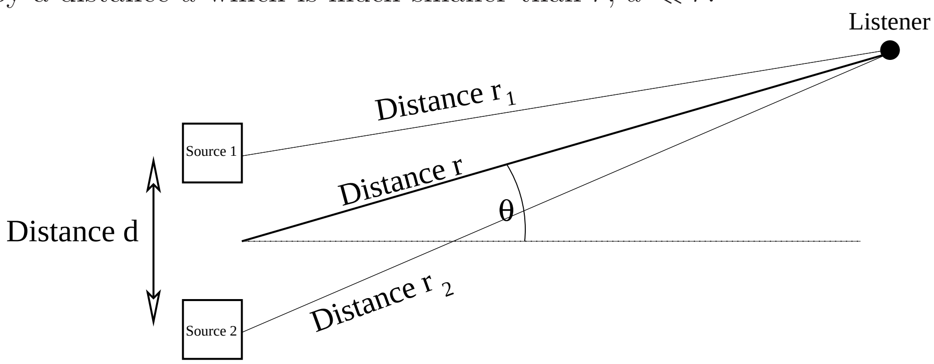
$$\begin{aligned} P_{\rm listener} &= P_{\rm speaker\ 1} + P_{\rm speaker\ 2} \\ &= A\left[\frac{1}{r_1}\sin\left(-2\pi f t + \phi + 2\pi\frac{r_1}{\lambda}\right) + \frac{1}{r_2}\sin\left(-2\pi f t + \phi + 2\pi\frac{r_2}{\lambda}\right)\right] \end{aligned} \tag{3.4}$$

and the intensity (loudness) is the square of this. In general this is some ugly expression. But very often, the "two speakers" will really be two parts of one object (such as a drumhead, the opening or bell of an instrument, the front plate of a violin, or whatever), in which case they are typically fairly close together. The listener is much further away. This makes the problem simpler in two ways:

- When the sources are close together and the listener is farther apart, we can replace $1/r_1$ and $1/r_2$ in front of the two expressions with $1/r$ the average of the separations (with little loss of accuracy). That means that the *magnitude* of the pressure from each source is the same (the sources separately sound to me to be of equal loudness).

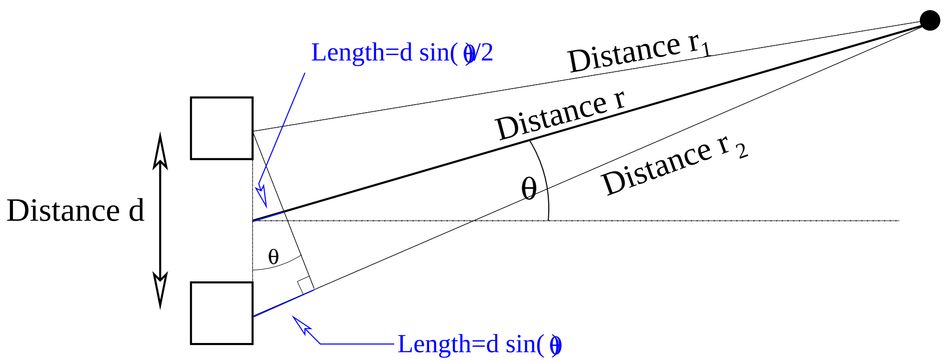
- When the sources are close together and the listener is farther apart, we may have to treat r_1 and r_2 inside the sine wave expressions more carefully, but we will be able to make some helpful approximations here as well.

So consider a listener at a distance r from the centrepoint of two speakers, which are separated by a distance d which is much smaller than r, $d \ll r$:



The figure shows the two distances from the two sources to the listener and defines them as r_1 and r_2. Since we know that r_1 and r_2 are almost the same, we can set them equal in the prefactors (the $1/r$ factors in front) in Eq. (3.4). But we *cannot* set them equal inside the sines, because there what matters is how large they are *in comparison to the wavelength* λ. That is, even if $r_1/r_2 \simeq 1$, if $r_1 - r_2$ is comparable to λ then the answers for these sines will be very different.

To find the differences between r, r_1, and r_2, we draw a line at right angles across the three lines:



There is an approximation here: I am pretending that the lines leading from the two sources to the listener are parallel. Of course they are not. But they are almost parallel if $r \gg d$ (r is much larger than d). If I am allowed this approximation, then the diagram shows that $r_1 = r - d/2$ and that $r_2 = r + d/2$. (The error in this approximation is proportional to $(d/r)^2$, and for most purposes it is not important.) Making these approximations, I find that r_1 is shorter than r by $(d/2)\sin\theta$ and r_2 is longer by $(d/2)\sin\theta$, as shown.

Making these approximations, the pressure the listener observes is

$$P = P_{\text{one source}} \left[\sin\left(2\pi \frac{r - d\sin\theta/2}{\lambda} - 2\pi f t + \phi\right) + \sin\left(2\pi \frac{r + d\sin\theta/2}{\lambda} - 2\pi f t + \phi\right)\right] . \tag{3.5}$$

To simplify this expression, define $\Phi = 2\pi r/\lambda - 2\pi f t + \phi$; then we can rewrite as

$$P = P_{\text{one source}} \left[\sin\left(\Phi - \frac{\pi d}{\lambda}\right) + \sin\left(\Phi + \frac{\pi d}{\lambda}\right)\right] . \tag{3.6}$$

Use

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B) \tag{3.7}$$

to turn this into

$$\begin{aligned} P &= P_{\text{one source}} \left[+ \sin(\Phi)\cos\left(\frac{\pi d \sin\theta}{\lambda}\right) - \cos(\Phi)\sin\left(\frac{\pi d\sin\theta}{\lambda}\right) \right. \\ &\qquad \left. + \sin(\Phi)\cos\left(\frac{\pi d\sin\theta}{\lambda}\right) + \cos(\Phi)\sin\left(\frac{\pi d \sin\theta}{\lambda}\right)\right] \\ &= P_{\text{one source}} \left[2\cos\left(\frac{\pi d\sin\theta}{\lambda}\right)\right] \, \sin\left(2\pi\frac{r}{\lambda} - 2\pi f t + \phi\right) . \end{aligned} \tag{3.8}$$

(In the last expression I rewrote out what Φ is.) Therefore the pressure is the same as the pressure from *one* source located at the midpoint between the two actual sources, times an extra factor of

$$\left[2\cos\left(\frac{\pi d \sin\theta}{\lambda}\right)\right] . \tag{3.9}$$

We should stop to examine this quantity. The intensity of the sound depends on the square of the pressure; therefore

$$I = I_{\text{one source}} \times 4\cos^2\left(\frac{\pi d\sin\theta}{\lambda}\right) . \tag{3.10}$$

For the case of two speakers which are close together ($d/\lambda \ll 1$), the argument of the cosine is small and the cosine is close to 1. Therefore the intensity is 4 times the intensity of a single speaker. The two speakers have strengthened each other, a surprising result. Physically, this occurs because the pressure produced by one speaker forces the other speaker to do more work when it pushes around the air.

For the case of two speakers which are far apart, so $d/\lambda > 1$, then the intensity is 4 times normal in some directions, but 0 times normal (silent) in other directions. Namely,

- Intensity maximum [4× one speaker] for $d \sin\theta/\lambda = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
- Intensity minimum [0× one speaker] for $d \sin\theta/\lambda = \ldots - \frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

This is because, for the first criterion, the argument of the cosine is a multiple of π so the cosine is ± 1. For the second criterion, the argument of the cosine is an odd $\pi/2$, so the cosine is zero. This means that there are patterns of loud and quiet spots. The places where the two sounds cancel are called *nodes*; the places where they add maximally are called *antinodes*. If d/λ is larger than 1, then if I average over directions the $4\cos^2(\ldots)$ averages to about 2, meaning that the intensity is twice the intensity of a single source–exactly what you would have expected from two sources.

Dipole sources

There are certain pairs of sources where the pressure produced by one is always high right when the pressure produced by the other is low. For instance, a speaker cone which is not in a housing behaves this way; the front side of the cone pushes the air forward (compresses it) right as the back side of the cone pulls the air behind it (decompressing it). A string moving through the air also behaves this way; the front side of the string compresses the air while the back side decompresses it. Such a pair of sources is called a *dipole source*.

In this case, the pressure far from the source adds up almost like in Eq. (3.5), except for the sign; since the sources are exactly out of phase (one pressure is high when the other is low), there is a minus sign between the two terms. Therefore Eq. (3.8) becomes instead

$$\begin{aligned} P_{\rm dipole} &= P_{\rm one\ source}\left[\sin\left(\Phi - \frac{\pi d}{\lambda}\right) - \sin\left(\Phi + \frac{\pi d}{\lambda}\right)\right] \\ &= P_{\rm one\ source}\left[+\sin(\Phi)\cos\left(\frac{\pi d \sin\theta}{\lambda}\right) - \cos(\Phi)\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)\right. \\ &\qquad \left. - \sin(\Phi)\cos\left(\frac{\pi d\sin\theta}{\lambda}\right) - \cos(\Phi)\sin\left(\frac{\pi d \sin\theta}{\lambda}\right)\right] \\ &= P_{\rm one\ source}\left[2\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)\right]\ \cos\left(2\pi\frac{r}{\lambda} - 2\pi f t + \phi\right) . \end{aligned} \tag{3.11}$$

If the sources are far apart, the behavior is about the same as before; all that changes is which spots are nodes and which are antinodes. If the sources are close together (the interesting case for dipole sources), then $\pi d/\lambda$ is small and we can make the approximation $\sin(x) \simeq x$ (x small):

$$P_{\rm small\ dipole} \simeq P_{\rm one\ source}\left[\frac{2\pi d\sin\theta}{\lambda}\right]\ \cos\left(2\pi\frac{r}{\lambda} - 2\pi f t + \phi\right) . \tag{3.12}$$

Therefore the intensity is modified by

$$I_{\text{small dipole}} = I_{\text{one source}} \times \frac{4\pi^2 d^2}{\lambda^2} \sin^2\theta\,. \tag{3.13}$$

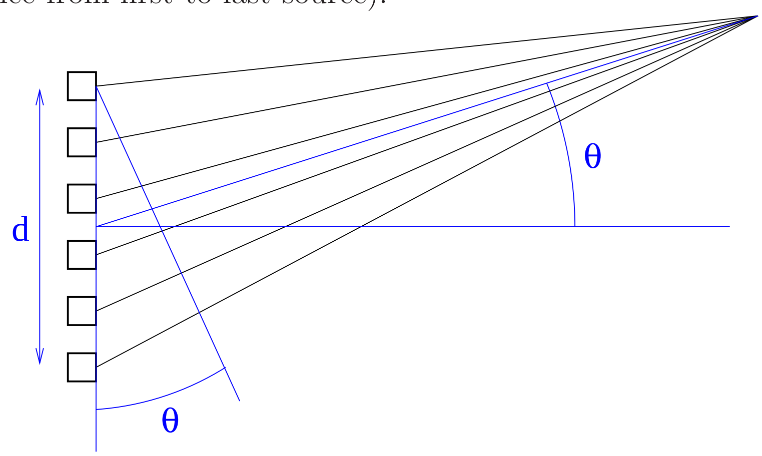
Often d is much smaller than λ; for instance, for a string, λ may be meters while d is the thickness of the string which is of order a millimeter. In this case, the cancellation between the two sources can knock orders of magnitude off the loudness of the source.

Note that it is more common to define the angles, for a dipole source, in terms of the line *connecting* the two sources. This change of definition for the angle θ just turns $\sin\theta$ into $\cos\theta$ in the above expression.

Chapter 4

Adding many waves; phasors

Next, consider sound arriving from several (say, 6) sources to a listener at distance $r \gg d$ (with d the distance from first to last source):



For the same reason as in the previous case, the difference between the shortest and longest path is $d\sin(\theta)$. That means the total pressure will be given by

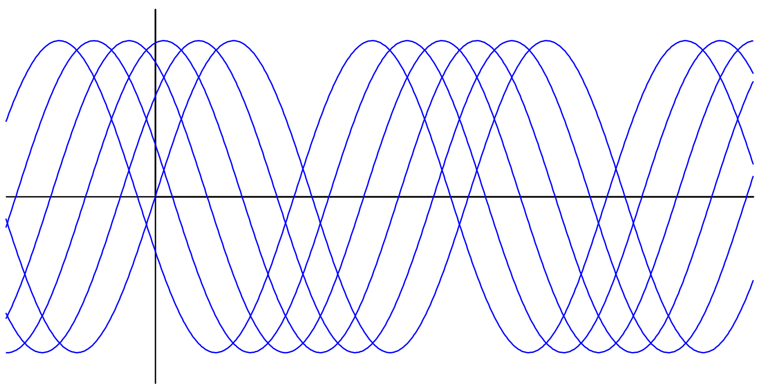
$$\begin{aligned} P_{\text{sources}} \; = \; P_{\text{one source}} \Big[\; & \sin\left(-2\pi\frac{5d\sin\theta}{10\lambda} + 2\pi\frac{r}{\lambda} - 2\pi f + \phi\right) \\ & + \sin\left(-2\pi\frac{3d\sin\theta}{10\lambda} + 2\pi\frac{r}{\lambda} - 2\pi f + \phi\right) \\ & + \sin\left(-2\pi\frac{1d\sin\theta}{10\lambda} + 2\pi\frac{r}{\lambda} - 2\pi f + \phi\right) \\ & + \sin\left(+2\pi\frac{1d\sin\theta}{10\lambda} + 2\pi\frac{r}{\lambda} - 2\pi f + \phi\right) \\ & + \sin\left(+2\pi\frac{3d\sin\theta}{10\lambda} + 2\pi\frac{r}{\lambda} - 2\pi f + \phi\right) \\ & + \sin\left(+2\pi\frac{5d\sin\theta}{10\lambda} + 2\pi\frac{r}{\lambda} - 2\pi f + \phi\right)\Big] \, . \end{aligned} \tag{4.1}$$

This looks completely horrendous. And when the source of a sound is some continuous medium or surface, we will really want to deal with the case where there are an infinite number of sources. So we had better develop some technique for dealing with this sort of calculation, preferably one which gives some clear way of visualizing what is going on.

4.1 Phasors

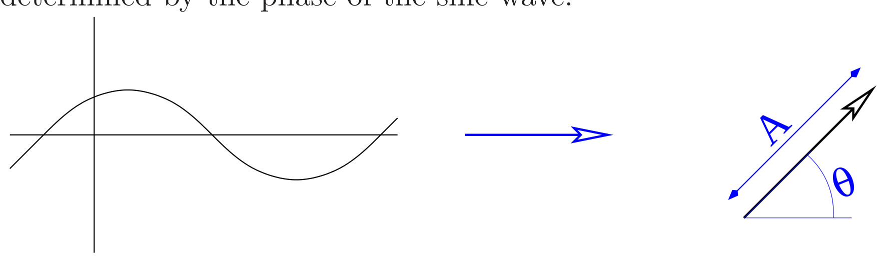
The method will involve an idea called a *phasor*. Since the idea is important, since it is not obvious, but since it *can* be understood intuitively, I will give three different explanations of how the approach works, in the hopes that one of them makes sense to you.

The problem we are trying to address is, how does one add up a whole bunch of sine waves with different phases, like



where each wave represents the pressure pattern (as a function of time) which one of the sound sources will produce at the listener's location. Though I have not considered it so far, in general each pressure wave can have a different height. We are only interested in sine waves with the same frequency.

The phasor method is to associate with each sinusoidal wave, a "vector" in a 2 dimensional space. The length of the vector is given by the amplitude of the sine wave; the direction of the vector is determined by the phase of the sine wave:

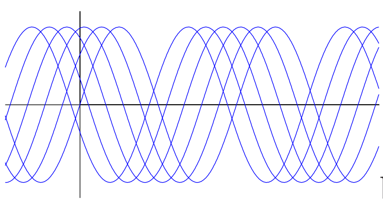
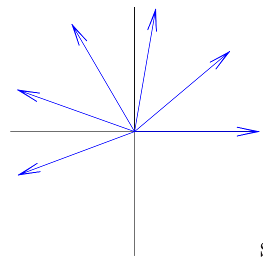
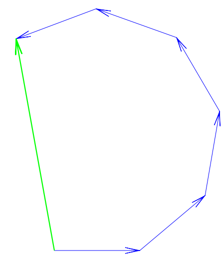


$A\sin(\theta + ft)$ Phasor length A angle θ

We associate such a 2-dimensional vector with each sinusoidal wave we want to add. Then we add the 2-dimensional vectors, *as vectors*, and it will tell us what sinusoidal wave we will get when we add the sinusoidal waves.

These funny 2-dimensional vectors are called *phasors*.

So, for instance, for the sinusoidal waves I showed earlier:

 become  sum to 

Then I reverse the process to figure out what sine wave this corresponds to. Most often, I am most interested in the amplitude (peak height) of the sinusoidal wave, and this is just given by the length of the phasor you get by taking the sum.

I *emphasize* that the horizontal and vertical components of a phasor DO NOT correspond to x and y position. They are two *abstract* directions.What the phasor method is doing is p

So now I have described the method, I should explain and justify it. I will do this 3 times; read at least the first two and pick your favorite way to look at it.

4.1.1 Decomposition into sines and cosines

Any sinusoidal wave can be broken into sine and cosine:

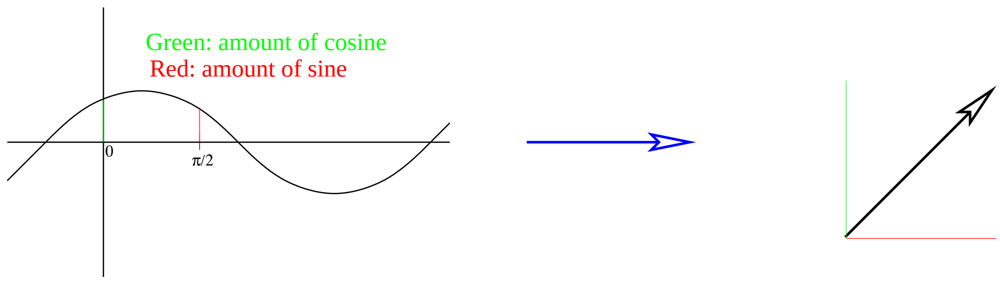
$$A\sin(\phi + 2\pi ft) = A\sin(\phi)\cos(2\pi ft) + A\cos(\phi)\sin(2\pi ft)\,. \tag{4.2}$$

Think of $A\sin(\phi)$ as the coefficient on $\cos(2\pi ft)$ the cosine wave and $A\cos(\phi)$ as the coefficient on $\sin(2\pi ft)$ the sine wave. If you have to add many sinusoidal waves, just do the same decomposition to each one:

$$\begin{aligned} & A\sin(\phi_1 + 2\pi ft) + B\sin(\phi_2 + 2\pi ft) + \ldots \\ = {} & A\left[\sin(\phi_1)\cos(2\pi ft) + \cos(\phi_1)\sin(2\pi ft)\right] + B\left[\sin(\phi_2)\cos(2\pi ft) + \cos(\phi_2)\sin(2\pi ft)\right] + \ldots \\ = {} & \cos(2\pi ft)\left(A\sin(\phi_1) + B\sin(\phi_2) + \ldots\right) + \sin(2\pi ft)\left(A\cos(\phi_1) + B\cos(\phi_2) + \ldots\right)\,. \end{aligned} \tag{4.3}$$

The phasor method is a graphical way of doing this, where you plot the *sine* component as the x axis and the *cosine* component as the y axis. Since vectors add in components (add all the x components to get the new x component, add the y components to get the new y component), this is a valid way of showing the way that sinusoids will add up.

If you have a sinusoidal wave, there is a quick way to figure out the size of the sine and cosine components. Measure the height of the wave at 0, where sine vanishes; this is the cosine component (which by our convention is the vertical component of the phasor). Measure the height at $\pi/2$, where cosine vanishes; this is the sine component (which by our conention is the horizontal part of the phasor):



One important thing about a phasor is that its length gives the peak height of the sinusoidal wave it is describing. We should check that this is true. Essentially this is true because the sine wave and the cosine wave are “orthogonal” functions in some sense; when you add them, the total size of the wave you get is the result of adding the coefficients of sine and cosine in quadratures, just like adding two vectors which are orthogonal gives a vector whose length squared is the sum of the squares of the two vectors’ lengths (Pythagorean theorem).

To see in detail that this is true, we need one fact about adding a sine to a cosine:

$$A\sin(2\pi ft) + B\cos(2\pi ft) = \sqrt{A^2+B^2}\ \sin\Big(2\pi ft + \arctan(B/A)\Big)\,. \tag{4.4}$$

If you believe this, you immediately see that the resulting wave is a sinusoid (a sine shifted by a phase), and its peak height is the square root of the sum of squares of the sine and cosine waves.

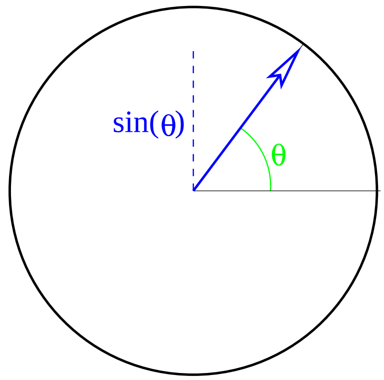
(You may want to check that Eq. (4.4) is really true. To do so, expand out the righthand side using $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$:

$$\begin{aligned}
& \sqrt{A^2+B^2}\ \sin\Big(2\pi ft + \arctan(B/A)\Big) \\
= \;& \sqrt{A^2+B^2}\,(\sin(\arctan(B/A))\cos(2\pi ft) + \cos(\arctan(B/A))\sin(2\pi ft)) \\
= \;& \sqrt{A^2+B^2}\left(\frac{B}{\sqrt{A^2+B^2}}\cos(2\pi ft) + \frac{A}{\sqrt{A^2+B^2}}\sin(2\pi ft)\right) \\
= \;& B\cos(2\pi ft) + A\sin(2\pi ft) \;\; \text{confirming claim}\,.
\end{aligned} \tag{4.5}$$

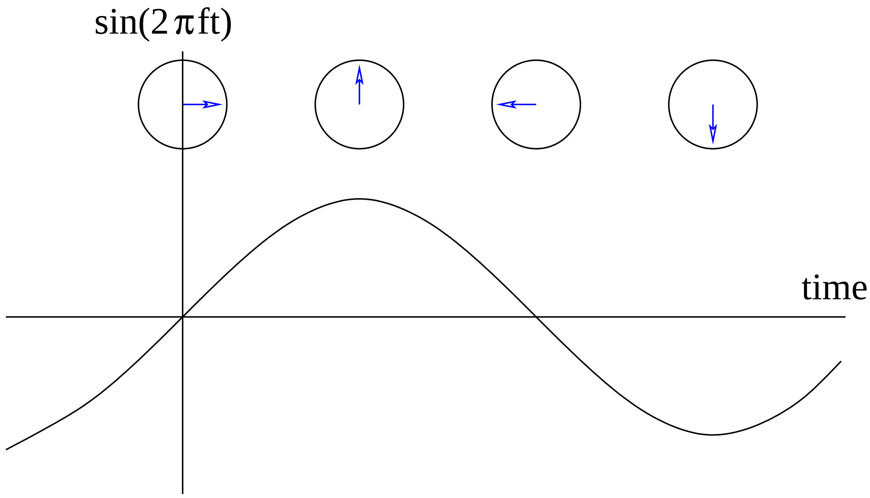
In the next to last step I used some facts about sines and cosines of inverse tangents.)

4.1.2 Components of a spinning arrow

Phasors work because sines and cosines actually *come about* as projections of a spinning arrow. Remember the “right” way to define the sine of an angle: draw the unit circle, start at the right, and go an angle θ around counterclockwise. The y position you end up at (the y component of the arrow from the centre of the circle to your position) is the sine of the angle θ:



The plot of $\sin(\theta)$ as a function of θ is just what you get as you go around the circle; so the plot of $\sin(2\pi ft)$ as a function of t is what you get if you make the arrow spin around in the circle by $2\pi f$ per unit time:



The phasors I have been describing are "the phasor at time 0" and the way the pressure (or whatever you are using the phasor to tell you about) will change with time is by such rotation by $(\pm)2\pi ft$.

So a sinusoidal wave can be described as the vertical projection of a spinning arrow. The length of the arrow is the maximum height and the starting direction of the arrow determines where the sinusoidal function goes through zero. So how do I add waves? Just by adding the arrows which are producing them! The reason is that, when you sum arrows as vectors, you are just adding components. And the y component is the height of the curve you want.

4.1.3 Complex exponentials

The following will only be useful to you if you are already comfortable with the mathematics of complex exponentials. If you are not, or if the following makes no sense to you, just ignore it.

Any sinusoidal wave can be expressed as the real part of a complex exponential:

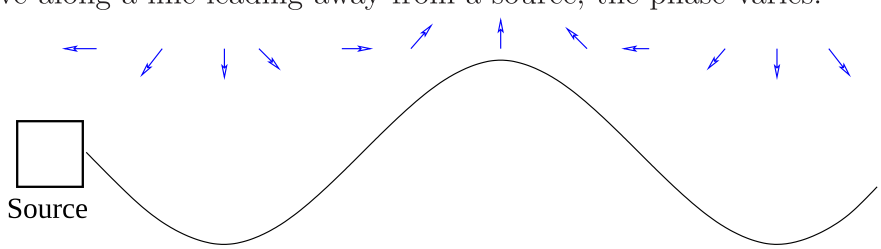
$$\cos(\phi + 2\pi f t) = \mathrm{Re}\, e^{i(\phi+2\pi f t)} = \mathrm{Re}\, e^{i\phi} e^{i\,2\pi f t}\,. \tag{4.6}$$

Here $i = \sqrt{-1}$ is the imaginary number, often written as j by engineers (physicists and mathematicians use i not j, so I will.)

Write all the sinusoidal waves we need to consider as the real parts of complex exponentials in this way:$e^{i2\pi f t}$ times some complex number. Plot these complex numbers in the complex plane. All the "2-component vectors" we have been discussing are really complex numbers; the two components are the real and imaginary parts. Addition in the complex plane is vector addition of 2-component vectors. Time evolution is "rotation" because I am multiplying a complex number by $e^{i\,2\pi f t}$, which is a pure phase; multiplication by a phase in the complex plane is rotation by that phase.

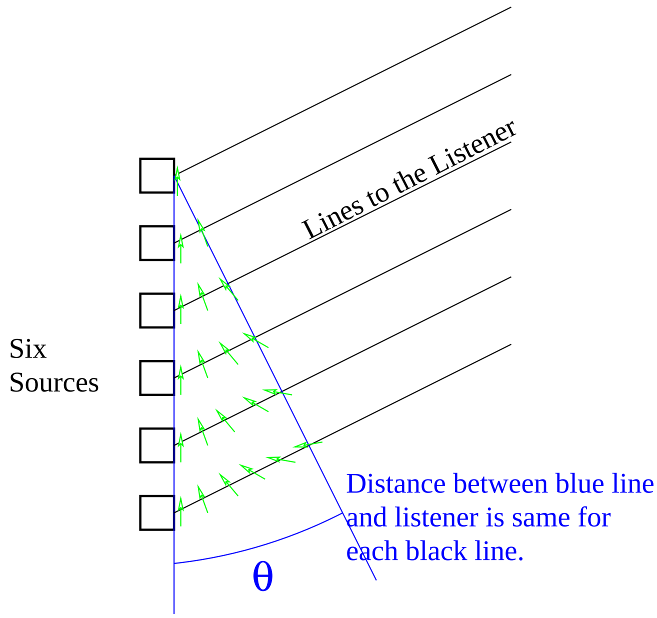
4.2 Application: a line of sources

Return to the problem of determining the loudness of a sound from 6 sources. Assume again that, at any moment, the sources each produce a sound with the same phase. But if I ask about the wave along a line leading away from a source, the phase varies:



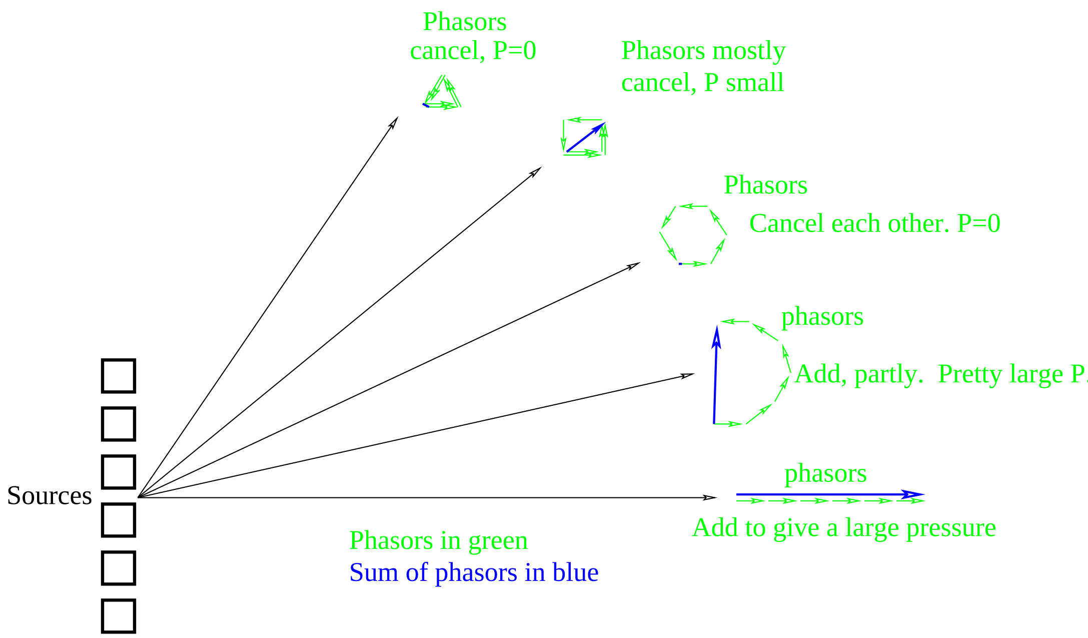
The reason, or way to understand, why the phasor's phase varies as you follow the wave away from the source is that, at each point along the wave, you are a different distance from the source. Therefore you are hearing the sound the source produced at a different time in the past (as emphasized in the last chapter). We saw that the phasor *rotates* as time goes by; so it rotates as a function of position along the line from source to listener. The rotation angle for a distance d is $2\pi d/\lambda$.

Now return to the problem at the start of the chapter; how do we add 6 sinusoidal waves from 6 sources? We see that the lines from each source to the listener are of different length. Therefore they accumulate phases which are different, by the amount of the difference in the lengths of the paths. We found that those differences were $2\pi d \sin(\theta)/5\lambda$, $2(2\pi d \sin(\theta)/5\lambda), \ldots$. At one moment in time, the phasors describing the sound at each point near the sources (zooming in near the sources and keeping track of the sound along each line from a source towards the listener) look like



where the phasors are in green. Of course the phasors continue to rotate as you go from the blue line (where I stopped plotting them) towards the listener; but since the distance to listener along each black line is the same past this point, they all rotate the same amount and it does not matter. To understand how much pressure I will observe at the listener, I have to add up the phasors along this blue line (where they are an equal distance from the listener). [Really, I have to add them at the listener, but as I just said, the answer will be the same except rotated, and I really care only about the length since its square tells me the intensity.]

The thing I need to know, to add those phasors, is the total angle difference between the first and last phasor. If you stand in front of the sources so $\theta = 0$, they are all at the same distance and so they add coherently; in this case you get a pressure which is 6 times the pressure from a single source. If you stand off-axis somewhat, the phasors do not all point the same way, and they will partly cancel. Pictorially, it will be something like this (this picture is for the case where the length of the line of sources d is a few times the wave length):



As the figure shows, if you are right in front of the sources, the phasors add constructively and you get a loud sound (large pressure, blue arrow). As you move away, the phasors are less alligned and they no longer add so completely. Then ,at some angle, they form a circle and cancel, giving no pressure (silence). When does this occur? For *THE SUM OF MANY* sources, it occurs (as the figure shows) roughly when the first and last arrows point in the *same* direction, or

Cancellation: maximum path length difference $d\sin(\theta) = \lambda$ 1 wave length

which is *different* from what we found for two sources, where it took a half wavelength. It is not enough for the first and last sources to cancel; that happens for the second-from-bottom path shown in the figure, but the sources in the middle are not canceled and they add up to something. We need the first source to cancel the middle source, the second source to cancel one down from the middle, and so on, so each source has something to cancel it. That requires that *half* the length difference be a half-wavelength.

For angles larger than the first angle to show perfect cancellation (for $d\sin\theta > \lambda$), the figure shows that the phasors stop canceling perfectly. But they cancel pretty well, and you do not get much intensity at those angles.

The case of many sources in a line is similar to having a continuous series of sources in a line. That is the limit where the arrows in the figure become more numerous but shorter. You can actually solve geometrically for the length of the sum (the blue arrow), without using any calculus. But if you know calculus it is much easier to do it that way.

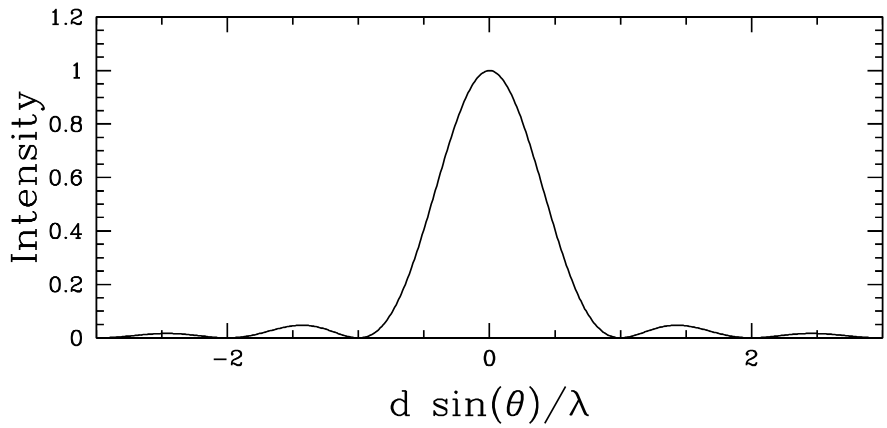
[If the following does not make sense to you, don't worry, just skip to the answer at the end.] Use the representation of phasors as complex numbers. Then replacing our line of 6 sources with a continuous line of sources, the extra phase a distance x down from the top of the line is $x \sin\theta$, and the sum of the phasors is given by

$$\begin{aligned} \text{Phasor sum} &= \int_0^d dx\ e^{i2\pi x \sin\theta/\lambda} \\ &= \frac{1}{2\pi \sin\theta/\lambda}\left(e^{i2\pi d \sin\theta/\lambda} - 1\right) \\ &= \frac{\sin(\pi d \sin\theta/\lambda)}{\pi \sin\theta/\lambda} \times \text{irrelevant phase}\,. \end{aligned} \tag{4.7}$$

Therefore the intensity varies with angle as [you can start reading here again]

$$I \propto \frac{\sin^2 \frac{\pi d \sin\theta}{\lambda}}{\frac{\pi^2 d^2 \sin^2\theta}{\lambda^2}}\,. \tag{4.8}$$

This function looks like

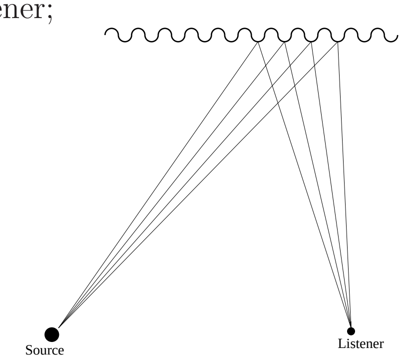


4.3 Sound reflection from a grating

I will make this section quite brief–the subject is probably confusing enough that a qualitative understanding is sufficient.

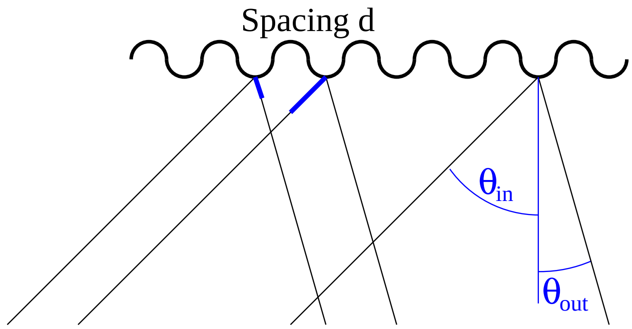
Consider a wall which has regular, periodic unevenness:

Consider bouncing a sound off of this wall from a source to a listener, for some general angle between source, wall, and listener;



As indicated, sound can bounce from the source to the listener along several paths–for instance, one path from the front of each "bump" on the wall. For the moment I will forget about sound reflecting from the "hollows" and only consider sound reflected from the "bumps"; we will come back to this soon.

The issue is that the path length for each of these ways sound can bounce from source to listener is different. That means that the sound will accumulate a different phase (be at a different point in the wave) for each reflected component. How do they add up? If the source and listener are far from the wall (the distance is large compared both to λ the wavelength and to d the spacing of bumps), then you can make similar geometic approximations to what we did last time:



The paths differ in length for two reasons. The right path takes longer to *reach* the wall, by $d \sin\theta_{\rm in}$. The left path takes longer to *leave* the wall, by $d \sin\theta_{\rm out}$. In the figure, these extra lengths are shown as fat, blue "bits" of the two sound paths. Therefore the path length difference between the two paths is

$$\text{Path length difference} = d(\sin\theta_{\rm in} - \sin\theta_{\rm out})\,. \tag{4.9}$$

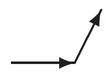
Now the key. If this path difference is exactly an integer number of wavelengths, then the sounds along the two paths will add in phase and we will get coherent reflection, leading to a loudness peak:

$$d(\sin\theta_{\rm in} - \sin\theta_{\rm out}) = \lambda \times (\ldots - 2, -1, 0, 1, 2, \ldots) \quad \Rightarrow \quad \text{Loudness peak.} \tag{4.10}$$

What if we are not at a multiple of λ? Then two neighboring paths will add somewhat out of phase. But there are more than two paths; there is a whole wall, with many many bumps. When we add these all together, the sound will cancel unless you are very close to perfect coherent addition between neighboring bumps on the wall.

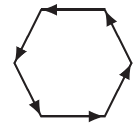
For instance, suppose that $d(\sin\theta_{\rm in} - \sin\theta_{\rm out})$ were $\lambda/6$ rather than exactly 0. Then two neighboring bumps would supply sine waves differing in phase by $\pi/3$. The phasors would be rotated by $\pi/3$ angle with respect to each other. This does *not* add up to zero. But when you add up the phasors for six bumps in a row, it *does* add up to zero:

Phasors for two bumps



Add up to something

But phasors for 6 bumps



Cancel completely

Because there is a whole string of bumps, even a small angle between the phasors for neighboring bumps leads to cancellation when you sum over the reflection from many bumps. Therefore sound is only reflected in the directions which satisfy the condition Eq. (4.10).

There is also sound reflected from the *grooves* in the pattern. I can add up the sound reflected from all the grooves (or from all the sides, etc) and I will find the same condition. The only question left is, does the sound from the grooves add to or cancel the sound from the bumps?

As long as the grooves are really behind the bumps, there is an extra path length (comparing a path using a bump to the path using the neighboring groove) because of the depth of the groove. In general this means that the sound from the grooves and from the bumps adds in some incoherent way. So, yes, there will be sound from both and in general it will add incoherently, meaning that the sound directions with "peaks" which we just learned about will really happen. But if the grooves are extremely shallow (the wall is almost flat, with very slight undulation)

then they will actually be about $\pi/2$ out of phase for the case $d(\sin\theta_{\rm in} - \sin\theta_{\rm out}) = \lambda$, and will therefore cancel. Their failure to cancel depends on the grooves being of order $d/4$ deep, which is enough to give an extra path length of order $d/2$ for the "groove" reflection. Also note that if, instead of bumps and grooves, we had alternating reflective and absorptive material, then diffraction would again happen, because there would only be sound from the reflective places.

[This is another advanced comment you don't have to understand. In general, a wall is not periodic; it is more complicated. If it is fairly close to planar, then you can Fourier transform its "bumpyness" and each Fourier component will act like a grating; the component with wave vector k has periodicity $2\pi/k$ which you should use for d to find out what angles it causes reflection; the amplitude of the Fourier component determines the strength of this reflection.]

What the above means is that, if you want to "break up" reflections and have them go in every direction, you have to make an "unpatterned" wall with unevennesses on every length scale and no repetition.

Chapter 5

Standing waves and pipe resonances

5.1 Uniform pipe

For a general overview of the material in this chapter, read chapter 21 of the Physics 224 notes. Since these are complete at the descriptive level, here I will just show the equations which underlie things.

First, consider standing waves in the air or in a cylindrical (uniform width) tube. A standing wave is just the sum of a wave moving to the left and a wave moving to the right:

$$\begin{aligned} P &= \frac{P_0}{2}\left(\sin\left[2\pi\frac{r}{\lambda} - 2\pi f t\right] + \sin\left[-2\pi\frac{r}{\lambda} - 2\pi f t\right]\right) \\ &= \frac{P_0}{2}\left(\sin\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi f t) - \cos\left[2\pi\frac{r}{\lambda}\right]\sin(2\pi f t)\right. \\ &\qquad \left. + \sin\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi f t) + \cos\left[2\pi\frac{r}{\lambda}\right]\sin(2\pi f t)\right) \\ &= P_0 \sin\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi f t)\,. \end{aligned} \tag{5.1}$$

Here I used $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$. Also note that, in general, $(2\pi\frac{r}{\lambda})$ could be shifted by a phase ϕ_r and $(2\pi f t)$ could be shifted by a phase ϕ_t (which just determine the location of the pressure zero and the time of maximum pressure).

We can determine the air velocity analogously. The only difference is that the velocities of the two waves cancel when the pressures add (since the backwards moving wave has backwards velocity where the pressure peaks):

$$\begin{aligned} v &= \frac{v_0}{2}\left(\sin\left[2\pi\frac{r}{\lambda} - 2\pi f t\right] - \sin\left[-2\pi\frac{r}{\lambda} - 2\pi f t\right]\right) \\ &= \frac{v_0}{2}\left(\sin\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi f t) - \cos\left[2\pi\frac{r}{\lambda}\right]\sin(2\pi f t)\right. \\ &\qquad \left. - \sin\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi f t) - \cos\left[2\pi\frac{r}{\lambda}\right]\sin(2\pi f t)\right) \\ &= -v_0 \cos\left[2\pi\frac{r}{\lambda}\right]\sin(2\pi f t)\,. \end{aligned} \tag{5.2}$$

The air motion is maximum at a location $\lambda/4$ away from where the pressure is maximum, and the air moves fastest $1/4f$ later than when the pressure peaks.

It is straightforward to check that these results for P and v actually satisfy the equations we found in the first lecture:

$$
\begin{aligned}
\rho\frac{dv}{dt} &\stackrel{?}{=} -\frac{dP}{dx} \\
-2\pi f\rho v_0\cos\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi ft) &\stackrel{?}{=} -\frac{2\pi}{\lambda}P_0\cos\left[2\pi\frac{r}{\lambda}\right]\cos(2\pi ft)
\end{aligned}
\tag{5.3}
$$

which is true because $P_0/v_0 = c_s\rho = f\lambda\rho$. The other condition $dP/dt = -\gamma P_{\rm atmos}dv/dx$ follows similarly.

The issue of the spatial appearance of this pattern and how it fits inside of a tube is well treated in the Phys 224 notes and will not be repeated here. The answer is

$$
f_{\rm resonant} = \frac{c_s}{4L}\times(1,3,5,\ldots)\,, \qquad \text{Open-closed tube}\,,
$$

and

$$
f_{\rm resonant} = \frac{c_s}{2L}\times(1,2,3,\ldots)\,, \qquad \text{Open-Open tube}\,.
$$

Incidentally, a closed-closed tube would have the same resonant frequencies as an open-open tube.

5.2 Resonances in a wine bottle

The wine bottle is more challenging. Here there are two "tubes": the body of the winebottle, and the neck. The body has cross-sectional area A_1 and length l, the neck has length d and cross-sectional area A_2. The bottom is closed, the top is open. Therefore the pressure must have an antinode (velocity node) at one end of the body. The pressure must also have a node at the opening of the neck. In other words, inside the body, measuring position *from the bottom of the bottle*, the pressure must be

$$
P_{\rm body} = P_1\cos\left(\frac{2\pi x}{\lambda}\right)\sin 2\pi ft \tag{5.4}
$$

where x is the distance from the bottom. Inside the neck, *measuring from the opening*, the pressure must be

$$
P_{\rm neck} = P_2\sin\left(\frac{2\pi y}{\lambda}\right) \tag{5.5}
$$

where y is the distance from the opening.

Where the neck and body meet, the pressures must equal, and the air flows must also be equal. Equal pressures means

$$
P_1\cos\left(\frac{2\pi l}{\lambda}\right) = P_2\sin\left(\frac{2\pi d}{\lambda}\right)\,. \tag{5.6}
$$

Equal air flows means

$$A_1 P_1 \sin\left(\frac{2\pi l}{\lambda}\right) = A_2 P_2 \cos\left(\frac{2\pi d}{\lambda}\right) . \tag{5.7}$$

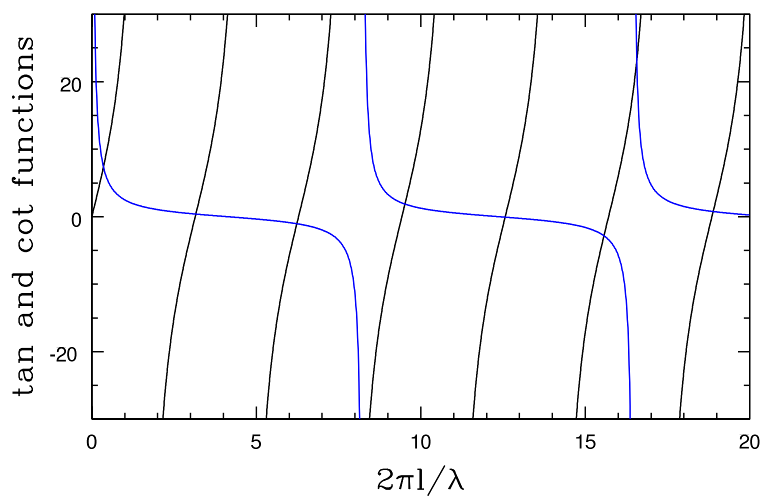
We want to know the wavelength λ. To find it, take the ratio of the two equations:

$$\frac{A_1 P_1 \sin(2\pi l/\lambda)}{P_1 \cos(2\pi l/\lambda)} = \frac{A_2 P_2 \cos(2\pi d/\lambda)}{P_2 \sin(2\pi d/\lambda)} \tag{5.8}$$

or

$$A_1 \tan \frac{2\pi l}{\lambda} = A_2 \cot \frac{2\pi d}{\lambda} . \tag{5.9}$$

One way to find the solutions to these equations is to graph each side and see where they cross. Here is the answer for the case $A_1/A_2 = 20$ and $l/d = 2.6$, which is roughly correct for the wine bottle:



We can also find the approximate location of the lowest frequency or longest wavelength minimum, which is the one you hear when you blow on the bottle. We do this by treating λ to be large (low frequency) and expanding the tangent and cotangent functions, $\tan(\theta) \simeq \theta$ and $\cot(\theta) \simeq 1/\theta$:

$$\begin{aligned} \frac{A_1\, 2\pi l}{\lambda} &\simeq \frac{A_2 \lambda}{2\pi d} , \\ \frac{1}{\lambda^2} &\simeq \frac{A_2}{4\pi^2 d\, l A_1} , \end{aligned} \tag{5.10}$$

which, using $f = c_s/\lambda$, gives

$$f \simeq \frac{c_s}{2\pi}\sqrt{\frac{A_2}{d\, l A_1}} = \frac{c_s}{2\pi}\sqrt{\frac{A_{\rm neck}}{d_{\rm neck} V_{\rm body}}} \tag{5.11}$$

where I used that the volume of the body is its length times its area.

5.3 Conical pipes

A qualitative description of the conical pipe is given in the Phys 224 notes, chapter 21. There it is explained how to think about the result: as the sound wave propagates towards the tip of the cone, it reflects before reaching the tip, approximately $\lambda/4$ away from the tip. Since the reflection is from a wide to a narrow tube, the pressure keeps rising as you go to the tip, but most of the sound intensity does not explore the tip. The pipe acts effectively like an open-closed tube which is $\lambda/4$ shorter than it "should be."

In other words, the longest wavelength to fit in the pipe should be $4L_{\rm eff}$, with $L_{\rm eff}$ the effective length given this "shortening" effect. But $L_{\rm eff} = L - \lambda/4$. The solution is

$$\lambda = 4(L - \lambda/4) \qquad \rightarrow \qquad 2\lambda = 4L\,, \qquad \lambda = 2L\,. \tag{5.12}$$

Similarly, the next resonance has

$$\lambda = \frac{4}{3}(L - \lambda/4) \qquad \rightarrow \qquad \frac{4}{3}\lambda = \frac{4}{3}L\,, \qquad \lambda = L \tag{5.13}$$

and so on. The wavelengths with resonances turn out to be precisely those for an open-open pipe, even though the behavior is like an open-closed pipe. This effective shortening of the pipe leads to a frequency series

$$f = \frac{c_{\rm s}}{2L} \times (1, 2, 3, \ldots)\,, \qquad \text{Conical pipe}\,. \tag{5.14}$$

Now we do the math. This will be a bit advanced, so don't worry if the following makes no sense. Combine the equations Eq. (1.18) by taking the space derivative of the first and the time derivative of the second:

$$\frac{d^2 v}{dxdt} = -\frac{1}{\rho}\frac{d^2 P}{dx^2} \qquad \text{and} \qquad \frac{d^2 P}{dt^2} = -\gamma P_{\rm atmos}\frac{d^2 v}{dxdt}\,. \tag{5.15}$$

(Derivatives commute with each other.) Eliminate $d^2v/dxdt$ and use that $\gamma P_{\rm atmos}/\rho = c_{\rm s}^2$. The result is that the pressure P must satisfy the wave equation

$$\frac{d^2}{dt^2}P = c_{\rm s}^2 \nabla^2 P\,. \tag{5.16}$$

It is easiest to work in spherical coordinates with the tip of the cone as the origin. In these coordinates $\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$. The most general solution with frequency f and "wavelength" $\lambda = c_{\rm s}/f$ turns out to be

$$P = \sin(2\pi f t) \times \left(A\frac{\sin(2\pi r/\lambda)}{r} + B\frac{\cos(2\pi r/\lambda)}{r} \right)\,. \tag{5.17}$$

You can (should?) verify that this really solves the wave equation. For those who like names, these are spherical Bessel functions (technically the sine is the Bessel function and the cosine is the Neumann function). Only the sine gives sensible (finite) behavior at the tip of the cone. So the pressure behaves like [Derivation ends, presentation of answers you might want to know begins]

$$P = P_0 \sin(2\pi f t) \frac{\sin \frac{2\pi r}{\lambda}}{r} \tag{5.18}$$

and the velocity behaves like its derivative,

$$v = \frac{P_0}{Z} \cos(2\pi f t) \left[\frac{\cos \frac{2\pi r}{\lambda}}{r} - \frac{\lambda}{2\pi r} \frac{\sin \frac{2\pi r}{\lambda}}{r} \right] . \tag{5.19}$$

It is not obvious that this expression for air velocity goes to zero at $r = 0$ (the tip of the cone) but it does.

The zeros of the pressure function are wherever the sine function has its zeros. These are *exactly* the places where the sine function we needed for an open-open cone had its zeros. The only difference is that, because of the $1/r$ bit, the would-be zero at $r = 0$ is actually a maximum–so this is an antinode, as needed at a closed end of a tube.

Chapter 6

End corrections and radiation

In the last section we made two approximations about the end of a pipe: first, that the pressure is zero there, and second, that all of the sound reflects from the end. Neither approximation is exact. Here we explore the corrections to these approximations.

6.1 Sound wave and emitted power

Suppose a pipe of radius a terminates. There is a sound wave of frequency f and wavelength $\lambda = c_{\rm s}/f$ inside the pipe, reflecting from the opening. Call the peak airspeed due to the *forward sound wave only* $v_{0,\rm in}$. At the level of approximation we used in the last section, the airflow out the mouth of the pipe is $2v_{0,\rm in}A = 2\pi a^2 v_{0,\rm in}$ (since the opening area A is πa^2). This flow must create some kind of air flow outside the pipe. Our goal is to determine the airflow *outside* the pipe, to figure out how much sound power is carried away.

Choose coordinates where the centre of the opening of the tube is at 0 and the tube is along the z axis. Work in spherical coordinates. More than a few times a away from the pipe, it is "small" and we can approximate the sound wave going out from the pipe as being spherical, that is, of going out equally in all directions, see Fig. 6.1.

In the last section we found a solution in spherical coordinates for the pressure of a sound wave:

$$P(r,t) = P_0 \sin(2\pi f t) \times \frac{\sin(2\pi r/\lambda)}{r} \,. \tag{6.1}$$

There is another solution (you can check) which is

$$P(r,t) = P_0 \sin(2\pi f t) \times \frac{\cos(2\pi r/\lambda)}{r} \,. \tag{6.2}$$

We didn't use this solution in the last chapter because it "blows up" (becomes infinite) at $r = 0$ (the tip of the cone, in the last problem) and that did not correspond with how sound would act at the tip of a cone, since there was nowhere for air to "go". But for the current problem there *is* somewhere for air to "go," since at $r = 0$ there is a tube which is supplying

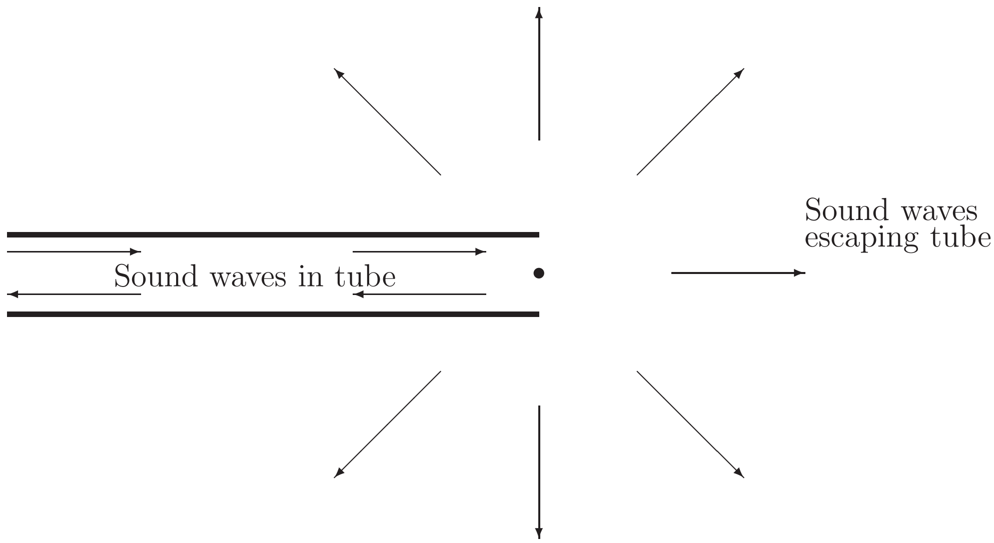


Figure 6.1: Waves traveling along a pipe, and outgoing waves from the mouth of the pipe going out in all directions. The heavy dot is our choice of "origin" for a coordinate system.

a flow of air. Instead, we should make sure that we choose a solution in which the sound wave is only moving away from the opening of the tube–at least at large distances. That means that we want a solution where the velocity is always in phase with the pressure (v is directed radially outward where the pressure is high) at large distances from the opening. (After all, we want to know what happens when sound is bouncing back and forth inside a tube, not what happens when sound comes in towards the tube from far away.)

The solution we want is the one which, at large distance, looks like $P \propto \sin(2\pi ft - 2\pi r/\lambda)$, just like the solution from the old days. This requires a special combination of the two solutions above:

$$P(r,t) = A\frac{\sin(2\pi ft - 2\pi r/\lambda)}{r}, \tag{6.3}$$

$$v(r,t) = \frac{A}{Z}\left[\frac{\sin(2\pi ft - 2\pi r/\lambda)}{r} + \frac{\lambda}{2\pi r}\frac{\cos(2\pi ft - 2\pi r/\lambda)}{r}\right]. \tag{6.4}$$

If you are happy with calculus, you can check that the pressure expression solves Eq. (5.16) the wave equation and the v expression follows by applying Eq. (1.18). Otherwise, just take this to be "a solution someone figured out which has the right properties." I emphasize again that the only approximation we made here is that $r \gg a$; we also assumed that the sound wave goes out equally in all directions. Our experience with diffraction says this will be true if the "source" is small compared to a wavelength. In this case the "size" of the source is the size of the mouth of the tube. So we have to assume $a \ll \lambda$.

In a way our solution is very simple. The pressure as we move away from the pipe opening acts just like the solution we found for a plane wave, but with an extra $1/r$. This is just the right factor so that the wave intensity $I = P^2/Z$ falls away as $1/r^2$. This makes sense:

the area the sound wave goes out through grows as $4\pi r^2$, so the intensity has to die away as $1/r^2$ so that the total sound power emerging from the pipe is the same at every radius (sound energy does not "disappear" on its way out from the pipe opening).

We have two things left to do. We need to figure out how the pressure and air velocity in this outgoing wave are related to the ones inside the pipe. And we need to figure out how much sound power the sound wave carries. The second part is easy:

$$\text{Power emitted} = 4\pi r^2 \frac{\langle P^2 \rangle}{Z} = 4\pi r^2 \frac{A^2/r^2}{2Z} = 2\pi \frac{A^2}{Z} \,. \tag{6.5}$$

Of course, we still have to figure out what A is.

The expression for v is complicated. There is the term we expect, which looks just like the term for the pressure, $v = P/Z$. But then there is an extra term, going as $\cos(2\pi f t - 2\pi r/\lambda)$ instead of sine, and proportional to $1/r^2$. This term is 90° out of phase with the pressure–the same behavior as air velocity and pressure have inside a tube. That means that, close to the tube opening, the sound wave is more complicated. In particular, this extra piece has an air velocity which scales as $1/r^2$, not $1/r$. That is important, because it means that the total flow of air, $4\pi r^2 v_{\rm air}$, does *not* approach zero as you go towards $r = 0$ the opening of the pipe.

The reason this is important is that we have to figure out how intense the sound outside the pipe is. The key is that the air flowing out of the pipe has to equal the air leaving the pipe opening. The air leaving the opening of the pipe was found above to be $2\pi a^2 v_{0,\rm in}$. Now we see that the flow of air away from the mouth of the pipe is

$$\text{airflow} = \text{area} \times v_{\rm air} \simeq 4\pi r^2 \times \frac{A}{Z} \frac{\lambda}{2\pi r^2} = 2 \frac{A\lambda}{Z} \,. \tag{6.6}$$

That relates the pressure outside the pipe to the airflow inside the pipe. Of course, this "matching" works because the region very near the opening of the pipe does not have enough volume to "store" significant amounts of air. However this is only true for r small compared to a wavelength λ. But our solution for the outgoing wave is only valid for r at least a few times larger than a, so we are again relying on $a \ll \lambda$ for our approximations to be valid.

Now we can figure out the radiated sound power in terms of the sound power inside the pipe. We know from Eq. (6.5) how the emitted power is related to A. We can solve Eq. (6.6) for A:

$$A = \frac{Z}{2\lambda} \times \text{airflow}$$

and substitute:

$$\text{Power emitted} = 2\pi \frac{A^2}{Z} = \frac{\pi Z}{2\lambda^2} \times (\text{airflow})^2 \,. \tag{6.7}$$

We already saw that the airflow is $2\pi a^2 v_{0,\rm in}$. Therefore

$$\text{Power emitted} = \frac{2\pi^3 Z a^4 v_{0,\rm in}^2}{\lambda^2} \,. \tag{6.8}$$

Now we need to find how much power is hitting the end of the tube. The intensity of the forward moving wave inside the tube is $I = Zv_{0,\rm in}^2/2$ and the power is the area πa^2 times this:

$$\text{Power incident} = \pi a^2 \frac{Zv_{0,\rm in}^2}{2} \,. \tag{6.9}$$

The ratio is

$$\frac{\text{Power emitted}}{\text{Power incident}} = \frac{2\pi^3 Z a^4 v_{0,\rm in}^2}{\lambda^2} \frac{2}{\pi a^2 Z v_{0,\rm in}^2} = \frac{4\pi^2 a^2}{\lambda^2} \,. \tag{6.10}$$

In other words,

$$\boxed{\text{Emission fraction} = \frac{4\pi^2 a^2}{\lambda^2} = \frac{a^2}{(\lambda/2\pi)^2} = \frac{4\pi^2 a^2 f^2}{c_{\rm s}^2} \,.} \tag{6.11}$$

This answer has two important features:

1. The emission fraction scales as a^2. That means that a wide opening is more efficient at emitting sound than a narrow opening. As we will see, it turns out that wind instruments rely on most of the sound reflecting back into the instrument for their operation. Therefore all wind instruments have relatively narrow openings–the tube is much longer than the opening is wide. The widest openings are in the brass instruments, which are indeed the loudest instruments. The sound intensity inside the bore of an oboe, a saxophone, or a trombone may be comparable. But the oboe opening is small, the saxophone opening is medium, and the trombone opening is wide; so the loudness we hear–what is emitted from the instrument–goes in this order: the oboe is soft, the saxophone is medium, and the trombone is loud.

2. The emission fraction scales as $1/\lambda^2$, or as f^2. Therefore, for a given instrument, the high pitched sounds escape more efficiently than the low pitched sounds. This has two effects. First, most wind instruments *can* be played louder (have a louder maximum dynamic) at higher frequencies than at lower frequencies. Second, the "tone color" or *timbre* of the sound is modified. The sound wave inside the instrument is often quite close to a sine wave, but with relatively weak harmonics (for instance, a loud 200 Hertz sound with softer 400, 600, and 800 Hertz sounds). But these harmonics are emitted from the instrument more efficiently than the fundamental. Therefore, the sound *we hear* is richer in harmonics than the sound inside the instrument. This is especially prominent in brass instruments because of the bell–a point we will come back to.

6.2 End correction

There is another interesting feature of the calculation we just performed. The pressure at the opening of the instrument is *not* zero. In fact, the pressure grows as $1/r$, as we saw.

Using the expressions we have above, the peak pressure at radius r is

$$P(r) = \frac{A}{r} = Z v_{0,\mathrm{in}} \frac{\pi a^2}{\lambda r} \,. \tag{6.12}$$

Furthermore, right near the opening the pressure is $\pi/2$ out of phase with the air velocity, that is, the pressure is varying as $\sin(2\pi f t)$ if the air velocity is varying as $\cos(2\pi f t)$. This is the same as the behavior for the standing wave inside the pipe.

Now the solution we found is really only valid far enough away from the pipe opening that the solution is "nearly spherical." As we argued, that means our solution only works for $r \gg a$. Qualitatively, our solution is sufficient to show us that, when the air is flowing outwards, there will be a positive pressure near the mouth of the pipe. We can also get a rough estimate for what the pressure right at the mouth of the tube should be, by making the approximation that we should set $r = a$ there. Under this *approximation*, we estimate

$$P(\mathrm{mouth}) \simeq P(r = a) = Z v_{0,\mathrm{in}} \frac{\pi a}{\lambda} \,. \tag{6.13}$$

It is enlightening to compare this with how fast the pressure rises as you go down the pipe away from the opening. Call the distance into the pipe d. Then the standard solution for the air velocity and pressure inside the pipe, *if* it satisfied the naive boundary condition $P = 0$ at the end, would be

$$v = 2 v_{0,\mathrm{in}} \cos(2\pi d/\lambda) \sin(2\pi f t) \,, \qquad P = 2 Z v_{0,\mathrm{in}} \sin(2\pi d/\lambda) \cos(2\pi f t) \,. \tag{6.14}$$

For small distances $\sin(2\pi d/\lambda) \simeq 2\pi d/\lambda$. Therefore the pressure a distance d inside the "ideal" pipe would be

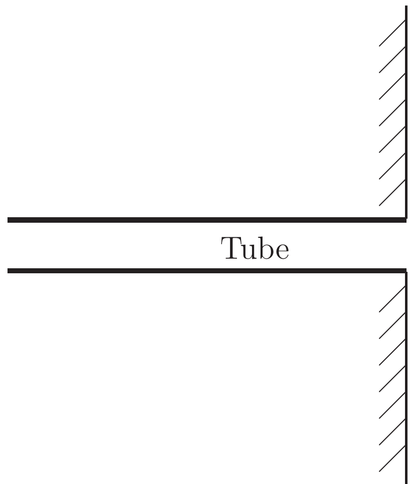
$$P_{\text{ideal pipe}}(d) = Z v_{0,\mathrm{in}} \frac{4\pi d}{\lambda} \,. \tag{6.15}$$

Our estimate of the pressure at the mouth is the same as the pressure would be if $P = 0$ at the opening, but the tube were $a/4$ longer than it really is.

Of course, this was only a rough estimate. A detailed calculation requires really solving the wave equation in the complicated geometry of the mouth of the pipe. This turned out not to be needed to figure out the power radiated in the last section, because we got lucky; the key quantity, the airflow, was r independent at small r. The pressure is not, though; it goes as $P \sim 1/r$. So the most important region for figuring out its behavior is $r \sim a$, which is exactly where the geometry is *NOT* simple and spherical approximations are *NOT* valid. The actual computation is *HARD* and I will not present it. It first appeared in 1948 in a paper by Levin and Schwinger (who later won a Nobel prize, though for other work). I have provided that paper on the webCT site if you want to look at it, but I don't think it will do you much good. The bottom line is:

> The pressure at the mouth of the tube is such that the tube "behaves as if" it were $0.613\,a$ longer than it really is.

There is also a calculation of this "end correction" for the case of a flanged pipe, meaning one which opens into a hole in a wall:



For this tube, the air moves out through a half-sphere of directions. The results, Eq. (6.3) and Eq. (6.4) for P and v are valid but they only apply for $2\pi r^2$ area instead of $4\pi r^2$ area. In order to match the airflows, we need A to be twice as big in this case. That means that the radiated intensity is 4 times as big. But it is only integrated over half as many directions ($2\pi r^2$ instead of $4\pi r^2$), and so the radiated power is

$$\text{Flanged: } \frac{\text{Power emitted}}{\text{Power incident}} = \frac{8\pi^2 a^2}{\lambda^2}\,. \tag{6.16}$$

The end correction is also different. The pressure needed to get this flow moving is larger. Again, it requires an elaborate calculation, though it turns out to be a little easier (Lord Rayleigh managed to do it in 1894). The result is:

> The pressure at the mouth of a **flanged** tube is such that the tube "behaves as if" it were $0.85\,a$ longer than it really is.

Let us quickly apply the equations for an open pipe to rewrite the resonant frequencies of open-open and open-closed pipes:

$$\text{Open-open: } f = \frac{c_{\text{s}}}{2(L + 2 \times 0.613\;a)} \times (1, 2, 3, 4, \ldots) \tag{6.17}$$

and

$$\text{Open-closed: } f = \frac{c_{\text{s}}}{4(L + 0.613\;a)} \times (1, 3, 5, \ldots)\,. \tag{6.18}$$

The extra factor of 2 for the open-open case is because *each* end acts as if it were $0.613\,a$ longer.

Chapter 7

Resonance

7.1 Another way to understand the bottle resonator

Let's think again about the wine bottle and its lowest resonant frequency. The key is to realize that the air in the neck and the air in the body are behaving very differently. The air in the neck is acting like a *mass*, while the air in the body is acting like a *spring*.

Call the dimensions of the bottle V the volume in the body, A the area of the neck, and d the length of the neck. The mass of the air in the neck is

$$m = \rho d A \, . \tag{7.1}$$

(We saw in the last chapter that d is the length of the neck plus end corrections equal to $0.61 + 0.85$ times the radius of the neck tube.) The resonance in the bottle consists of this air moving back and forth. (We saw in our explicit solution that the air velocity in the neck is almost constant.) The force on this air is determined by the overpressure inside the bottle (we saw when we explicitly solved for the resonant frequency that the pressure is approximately constant inside the "body" of the resonator):

$$ma = F = A \Delta P \, . \tag{7.2}$$

Here a is the acceleration of the air in the neck.

Call the position of the air in the neck x. That is, x is how far in or out the air in the neck of the bottle has moved. Then $a = \frac{d^2x}{dt^2}$. We need to figure out what ΔP is in terms of x. When the air moves forwards, it empties air out from the body of the bottle. The volume of air which is removed from the bottle is

$$\Delta V = Ax \, . \tag{7.3}$$

The loss of pressure is determined by

$$\Delta P = -\gamma P_{\rm atmos} \frac{\Delta V}{V} = -\frac{\gamma P_{\rm atmos} A}{V} x \, . \tag{7.4}$$

(See Section 1.3 for a discussion of why this is the change in the pressure.) Therefore the force on the air in the neck *because of the compression or rarefaction of the air inside the bottle* is

$$F = A\Delta P = -\frac{\gamma P_{\rm atmos} A^2}{V} x \equiv -Kx \,, \qquad K = \frac{\gamma P_{\rm atmos} A^2}{V} \,. \tag{7.5}$$

This looks the same as Hooke's law for a spring. Physicists define a "spring" as anything which pushes on a mass in proportion to its displacement; the further forward you move the air in the neck, the harder the air inside the bottle pulls it back in. The air inside the bottle is acting like an "air spring" and the quantity K is called the "spring constant."

With this definition, the position of the air in the bottle is determined by

$$m\frac{d^2x}{dt^2} + Kx = F_{\rm ext} \,, \tag{7.6}$$

where I allowed for a force $F_{\rm ext}$ pushing on the air at the opening of the bottle (for instance, from any sound waves in the room). Now figuring out how the air in the bottle behaves has turned into a freshman physics problem.

Suppose there is a sound in the room, of frequency f, which causes a force on the air in the bottle's neck of $F_{\rm ext} = F_0 \sin(2\pi f t)$. In what follows it will be convenient to work in terms of the *angular frequency* ω defined as $\omega = 2\pi f$. This is the quantity which keeps us from having factors of 2π all over. Using this notation, $F_{\rm ext} = F_0 \sin(\omega t)$. The way we solve the equation is to *guess* that

$$x = x_0 \sin(\omega t - \phi) \,,$$

which just says that we expect the air to oscillate at the same frequency as we push it. (What else would it do?) Here x_0 and ϕ are *unknowns* we will need to determine by making sure that Eq. (7.6) is satisfied.

So substitute our guess into the equation. Use that

$$\frac{d^2}{dt^2}\sin(\omega t) = -\omega^2 \sin(\omega t)$$

to find

$$(-m\omega^2 + K)x_0 \sin(\omega t - \phi) = F_0 \sin(\omega t) \,. \tag{7.7}$$

This will work provided that $\phi = 0$ and

$$x_0 = \frac{F_0}{K - m\omega^2} \,. \tag{7.8}$$

The behavior of x_0 is plotted in Fig. 7.1.

There are several features which bear note. First, there is a special frequency called the resonant (angular) frequency

$$\omega_0 \equiv \sqrt{\frac{K}{m}} \tag{7.9}$$

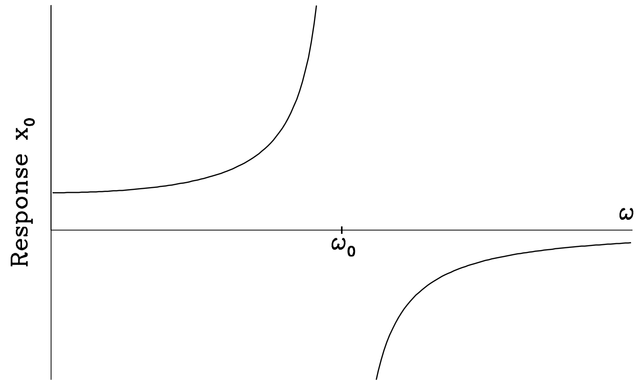


Figure 7.1: The amplitude x_0 of oscillation, before we account for damping. At the special frequency $\omega = \omega_0$, the size of the oscillations becomes *infinite*.

where the response becomes *infinite*. At lower frequencies, the displacement is in the same direction as the force. At higher frequencies, the displacement is in the opposite direction as the force. For the case of the bottle, this characteristic frequency is

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{\gamma P_{\rm atmos} A^2}{V} \frac{1}{\rho d A}} = \sqrt{\frac{\gamma P_{\rm atmos}}{\rho} \frac{A}{dV}} \, . \tag{7.10}$$

We recognize $\gamma P_{\rm atmos}/\rho = c_{\rm s}^2$ the squared speed of sound. Therefore

$$\omega_0 = c_{\rm s} \sqrt{\frac{A}{dV}} \, , \qquad f_0 = \frac{c_{\rm s}}{2\pi} \sqrt{\frac{A}{dV}} \, . \tag{7.11}$$

This is the same result we found in Eq. (5.11), by quite a different approach. So we see another feature of the resonant frequency; if you "push" on the resonance right at that frequency, the oscillations will grow without limit.

7.2 resonance with damping

Of course it should always worry us when we do a calculation of something and get an answer which is infinity. It must mean that we have left out some important effect. In this case, what we have left out is the effect of damping on the resonance. There are a few effects which cause the intensity of a sound wave to slowly die down as it bounces back and forth. We saw one in the last chapter: some of the sound is radiated into the room. There are other effects, which are frequently larger, which also cause energy losses when air oscillates in a narrow space like the neck of the bottle. For instance, the air "rubs" against the walls of the neck, causing friction losses. For the moment we will model this effect as an extra retarding force on the air which is proportional to its velocity (faster-moving air rubs harder on the walls and is slowed down more). Call the force per unit velocity of the air R. Then the position of the air must satisfy

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Kx = F_{\rm ext}\,. \tag{7.12}$$

Again we consider the case of a sinusoidal force $F_{\rm ext} = F_0 \sin(\omega t)$. Again the solution should look like $x = x_0 \sin(\omega t - \phi)$. Taking the derivatives,

$$-m\omega^2 \sin(\omega t - \phi) + R\omega \cos(\omega t - \phi) + K \sin(\omega t - \phi) = F_0 \sin(\omega t)\,. \tag{7.13}$$

Now use half-angle formulae:

$$\begin{aligned} \sin(\omega t - \phi) &= \cos(\phi)\sin(\omega t) - \sin(\phi)\cos(\omega t)\,, \\ \cos(\omega t - \phi) &= \sin(\phi)\sin(\omega t) + \cos(\phi)\cos(\omega t)\,. \end{aligned}$$

Therefore

$$\begin{aligned} &\Big([-m\omega^2 + K]\cos(\phi) + R\omega\sin(\phi)\Big)\sin(\omega t) + \Big([m\omega^2 - K]\sin(\phi) + R\omega\cos(\phi)\Big)\cos(\omega t) \\ =\ &F_0\sin(\omega t) + 0\cos(\omega t)\,. \end{aligned} \tag{7.14}$$

This only works if the coefficients on $\sin(\omega t)$ and $\cos(\omega t)$ both agree. Therefore

$$R\cos(\phi) + (m\omega^2 - K)\sin(\phi) = 0 \qquad \Rightarrow \qquad \phi = \arctan\frac{\omega R}{K - m\omega^2}\,, \tag{7.15}$$

and (using the above and doing some ugly but mindless algebra)

$$x_0 = \frac{F_0}{\sqrt{(K - m\omega^2)^2 + \omega^2 R^2}}\,. \tag{7.16}$$

This behavior is illustrated in Fig. 7.2

Some comments are in order:

- For small ω, the phase angle is 0, which means that the air moves back as it is pushed back and moves forward as it is pushed forward. The amplitude of the motion is $x_0 = F_0/K$. This is exactly the behavior expected for "pushing on a spring." The mass and resistance are irrelevant at low frequency, only the spring constant K determines the behavior.

- For large ω, the phase angle is 180°, meaning that when the force is forwards, the air position is backwards and vice versa. The amplitude of the motion is $x_0 = F_0/\omega^2 m$. This is the behavior you expect from "pushing on a mass." The phase is the most confusing part, so it bears clarifying. As the mass moves forward, the force has to switch to pushing it in, to get it to stop moving and turn around. Once the mass is moving inward, the force must switch to pulling it out. Alternatively, look at how the equations would be solved if there were only the mass term; $-\omega^2 m x_0 = F_0$, so $|x_0| = F_0/m\omega^2$ but x_0 is of opposite sign as F_0.

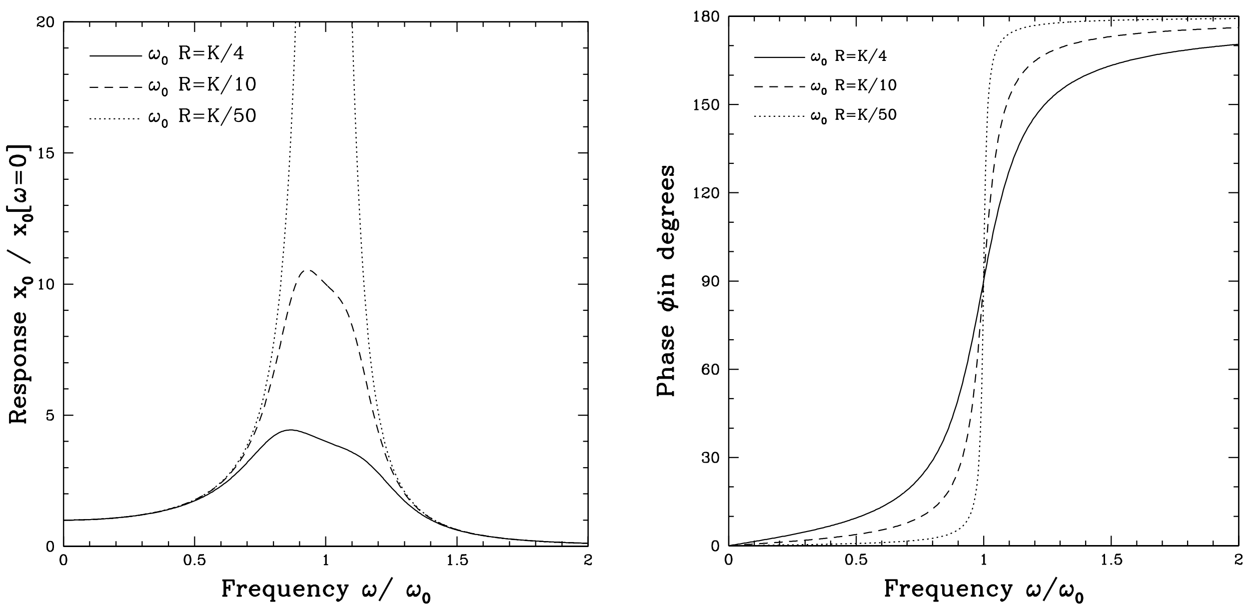


Figure 7.2: Response of a damped oscillator to a sinusoidal force, for three values of the damping term: $\omega_0 R = K/4$, $\omega_0 R = K/10$, and $\omega_0 R = K/50$. Left: amplitude x_0 of oscillations. Right: phase ϕ of oscillations.

- For ω close to ω_0, the resistance plays a role. It limits how large the oscillations can grow. The range of frequencies where resistance is important is centered on ω_0 and has a width of about $\omega_0 \times (\omega_0 R/K)$ on either side of ω_0. The phase shifts from "springlike" to "masslike" over this range, and in this range the response is large but smaller than it would have been without resistance.

- Right at $\omega = \omega_0$, the external force is perfectly in synch with the resonant frequency of the oscillator. The spring-mass system would bounce with "infinite" height if it were not for the resistance. The result is that the mass and spring constant effectively cancel each other's effects, and the force is only working against the resistance. The phase $\phi = \pi/2 = 90°$ means that the air is moving forward when the force pulls it forwards and is moving backwards when the force is pushing it backwards. The amplitude is $|x_0| = F_0/\omega R$, or if you prefer, $v = F_0/R$. This is what you would expect if only the resistance term were there.

7.3 Free oscillation, ringdown

What if the air in the bottle is already moving and pressurized, and I "let it go" to see how it will bounce back and forth? We can guess that, if $R = 0$, then we have an oscillator which should bounce back and forth sinusoidally with frequency $f_0 = \omega_0/2\pi$. What happens when there is resistance?

First I will develop an intuitive answer. Then we will see how to do the math.

7.3.1 Intuitive discussion

Suppose the resistance R is small. Then roughly we expect the air to oscillate as it would with no resistance at all,

$$x = x_0 \sin(\omega_0 t)\,. \tag{7.17}$$

However this cannot be quite right because the resistance eats up energy. The energy in the oscillator is

$$E = \frac{m}{2}v^2 + \frac{K}{2}x^2\,. \tag{7.18}$$

We know the $mv^2/2$ term from standard mechanics. The $Kx^2/2$ term is the "compression" term we learned about in Chapter 2. The other way to see that it is right is to notice that *change* in energy is force times distance. Since force goes as Kx, the force times distance will be Kx^2 but with an extra factor of 1/2 for the same reasons we saw in Chapter 2.

We can average each of these terms. The average of x^2 is $x_0^2/2$ (because we are averaging the square of a sine wave). If $x = x_0 \sin(\omega_0 t)$ then $v = \omega_0 x_0 \cos(\omega_0 t)$ and so $\langle v^2 \rangle = \omega_0^2 x_0^2/2$. The energy is therefore

$$E = \frac{m\omega_0^2 + K}{4}x_0^2 = \frac{K}{2}x_0^2\,, \tag{7.19}$$

where I also used that $\omega_0^2 = K/m$.

So now we know how much energy is associated with the oscillation of the resonator. Now we need to figure out how fast energy is lost. The term Rdx/dt in the oscillator equation is a force term, and power (change of energy) is force times velocity. Therefore the rate at which energy is dissipated by the resistance term is

$$\frac{dE}{dt}[\text{resistance}] = (-Rv) \times v = -Rv^2\,, \qquad \rightarrow \qquad \text{Time averaged } \frac{dE}{dt} = -R\omega_0^2 \frac{x_0^2}{2}\,. \tag{7.20}$$

In finding the time average I did the same thing that I did to find the time average of v^2 previously.

The key is that x_0^2 is related to the energy: $x_0^2 = 2E/K$ according to Eq. (7.19). Substituting this in, we find out that

$$\frac{dE}{dt} = -R\omega_0^2 \frac{E}{K} = \frac{-R\omega_0^2}{K}E\,. \tag{7.21}$$

We have seen something like this before: an equation showing that the rate at which energy is lost is proportional to the energy itself. We already know what it will mean; the energy will decay away exponentially, with a time constant given by the inverse of the expression in front of E on the right hand side:

$$\tau = \frac{K}{R\omega_0^2} = \frac{m}{R}\,. \tag{7.22}$$

Therefore the amount of energy will decay away as $E = E_{\rm initial} e^{-t/\tau}$ with $\tau = m/R$.

Now the energy goes as the square of the amplitude $E \propto x_0^2$. Therefore, in the time it takes for the energy to decay by a factor of 4, the amplitude only decays by a factor of 2. It takes twice as long for x_0 to decay; or, the decay time for the amplitude is twice as long as for the energy. The behavior of x will therefore be approximately

$$x = x_0 \sin(\omega_0 t) e^{-t \times R/2m}\,, \tag{7.23}$$

sine wave oscillations times an exponential decay with falloff time $2m/R$.

It is common to compare the frequency scale ω_0, telling how fast the oscillator oscillates, to the frequency scale $1/\tau$, telling how fast the oscillator loses energy. The ratio is called the *quality factor* Q of the resonator

$$\omega_0 \tau \equiv Q \qquad \left(= \sqrt{\frac{Km}{R^2}} = \frac{K}{\omega_0 R} = \frac{\omega_0 m}{R} \right)\,. \tag{7.24}$$

Roughly, Q tells how many times the oscillator will bounce back and forth before it dies down (actually $Q/2\pi$ is the number of oscillations). Alternatively, it tells how narrow is the region of driving frequencies where the response is resistance dominated. In the last section, Q was the width in frequency ω over which the phase ϕ changes from near 0 to near π, divided by ω_0.

Physically, a high Q resonator is one which bounces back and forth many times before the resonance dies down. It has a very precise frequency where it will resonate. A low Q resonator has a wider range of frequencies where it acts basically the same. In playing an instrument with a high Q resonance, the performer will find that the instrument "wants" to play a very precise note and it is difficult to "pull" the note very far. A low Q resonance is easier to "pull," but in turn that makes it harder to keep in tune.

To clarify a little more and to explain why we define Q the way we do, go back to Eq. (7.16) and re-express it using Q:

$$x_0 = \frac{F_0}{\sqrt{(K - m\omega^2)^2 + \omega^2 R^2}} = \frac{F_0/K}{\sqrt{\left(1 - \frac{m}{K}\omega^2\right)^2 + \left(\frac{\omega R}{K}\right)^2}} \tag{7.25}$$

which, using $(m/K) = 1/\omega_0^2$ and $R/K = 1/(\omega_0 Q)$ becomes

$$x_0 = \frac{F_0/K}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\omega^2}{\omega_0^2 Q^2}}}\,. \tag{7.26}$$

We see that the response is set by the low frequency response F_0/K times a function which depends only on ω/ω_0 and Q. If we define $\omega' = \omega/\omega_0$ the ratio of frequency to the critical frequency, this is

$$x_0 = x_0[\omega = 0] \times \frac{1}{\sqrt{(1 - \omega'^2)^2 + \omega'^2/Q^2}}\,. \tag{7.27}$$

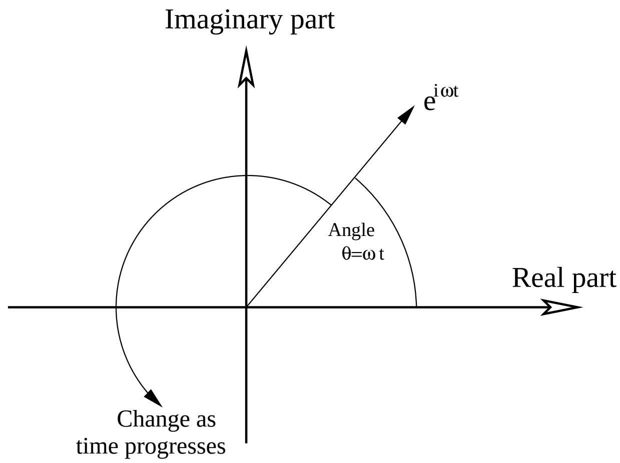


Figure 7.3: Illustration of how a complex exponential $e^{i\omega t}$ is a point of unit length in the complex plane. Its real and imaginary parts are the projections onto the real and imaginary axes, which are the cosine and sine of the angle ωt.

It is then clear that Q plays the role of telling how tall the peak at $\omega = \omega_0$ is, compared to the low frequency behavior: $x_0[\omega = \omega_0] = x_0[\omega = 0]/Q$.

7.3.2 Clever mathematical approach

There is a clever mathematical way to solve for the behavior of the oscillator. This approach has many other applications and so it is worth looking.

The key is to use the properties of the complex exponential $e^{i\theta}$. If θ is real, then this is a unit-length "vector" in the complex plane, with θ determining the angle from the real axis, see Figure 7.3. Therefore

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,, \qquad \Re e^{i\theta} = \cos(\theta)\,, \quad \Im e^{i\theta} = \sin(\theta)\,. \tag{7.28}$$

So instead of constantly messing with cosines and sines, we can do everything with complex exponentials. We just have to write "real part of" to specify that we really want the sinusoidal function.

What makes complex exponentials so useful is that the rules for taking derivatives are extremely simple.

$$\frac{d}{dt} e^{i\omega t} = i\omega e^{i\omega t}\,. \tag{7.29}$$

Derivatives turn into multiplicative factors. Calculus turns into algebra. Solving the driven oscillator equation becomes easy: write $F = F_0\cos(\omega t)$ as $F = \Re F_0 e^{i\omega t}$. Then the equation is

$$m\frac{d^2}{dt^2}x + R\frac{d}{dt}x + Kx = F_0 e^{i\omega t} \tag{7.30}$$

and guessing $x = x_0 e^{i\omega t}$ *with x_0 complex* results in

$$(-m\omega^2 x + Ri\omega x + Kx)e^{i\omega t} = F_0 e^{i\omega t} \tag{7.31}$$

and so

$$x = \frac{F_0}{K - m\omega^2 + i\omega R} \,. \tag{7.32}$$

What does it mean that x is complex? Remember that the behavior of x is determined by the real part of $xe^{i\omega t}$. If we write $x = x_r + ix_i$ then this is

$$\Re x e^{i\omega t} = x_r \cos(\omega t) - x_i \sin(\omega t) \,. \tag{7.33}$$

A better way to think about it is to write $x = x_0 e^{-i\phi}$. The angle ϕ is the phase difference between the driving force F and the response x; as ϕ increases, x falls further behind the driving force. The amplitude x_0 is the peak value the air position takes. If we rewrite $x = F_0/(K - m\omega^2 + i\omega R) = x_0 e^{-i\phi}$, it turns out that the values of x_0 and ϕ are the same as the ones we found earlier.

We can also define, as usual, the impedance

$$Z = \frac{F}{v} \tag{7.34}$$

the ratio of force to velocity. The velocity $v = dx/dt = i\omega x$ is now a complex quantity, so the impedance is complex as well:

$$Z = \frac{F}{i\omega x} = \frac{-iF_0/\omega}{\frac{F_0}{K - m\omega^2 + i\omega R}} = R + i\left(m\omega - \frac{K}{\omega}\right) \,. \tag{7.35}$$

The real part of the impedance is called the *resistance*. The imaginary part is called the *reactance*, often written as X:

$$Z = R + iX \,. \tag{7.36}$$

The inverse of the impedance is the *admittance*. It tells how much something will move when you push on it. Again it has real and imaginary parts:

$$\text{Admittance } Y \equiv \frac{1}{Z} = \frac{1}{R + iX} = G + iB \,, \qquad G = \frac{R}{R^2 + X^2} \,, \quad B = \frac{-X}{R^2 + X^2} \,. \tag{7.37}$$

To finish the name calling, G is often refereed to as the "conductance." Its role is to tell how much in-phase velocity there is when you push on something.

Now let us use this machinery to figure out the free response of the resonator. If there is no force on a resonator, it behaves according to Eq. (7.31):

$$(-\omega^2 m + i\omega R + K)x = 0 \,, \qquad \rightarrow \qquad 0 = K + i\omega R - \omega^2 m \,. \tag{7.38}$$

This is a quadratic equation. Its solution is

$$\omega = i\frac{R}{2m} + \sqrt{\frac{K}{m} - \frac{R^2}{4m^2}}\,. \tag{7.39}$$

The frequency we have found is complex! There is nothing wrong with that; just write $\omega = \omega_r + i\omega_i$ and stuff it into the exponential:

$$x = x_0 e^{i(\omega_r + i\omega_i)t} = x_0 e^{i\omega_r t - \omega_i t} = x_0 e^{i\omega_r t} e^{-\omega_i t}\,. \tag{7.40}$$

Taking the real part tells what the position really does:

$$x = x_0 \cos(\omega_r t) e^{-\omega_i t}\,, \qquad \omega_i - \frac{R}{2m}\,, \quad \omega_r = \sqrt{\frac{K}{m} - \frac{R^2}{4m^2}}\,. \tag{7.41}$$

The exponential rate of decay is just what we guessed by our energy balance methods. We missed a shift to the oscillation frequency; ω_0 is replaced by $\sqrt{\omega_0^2 - R^2/4m^2}$. We also found a cosine instead of a sine, but that depends on whether we take the coefficient x_0 to be real or complex; if we had assumed it was imaginary we would have found a sine.

Using the definition of Q we developed earlier, we find that the behavior of an oscillator when you let it bounce freely without driving is

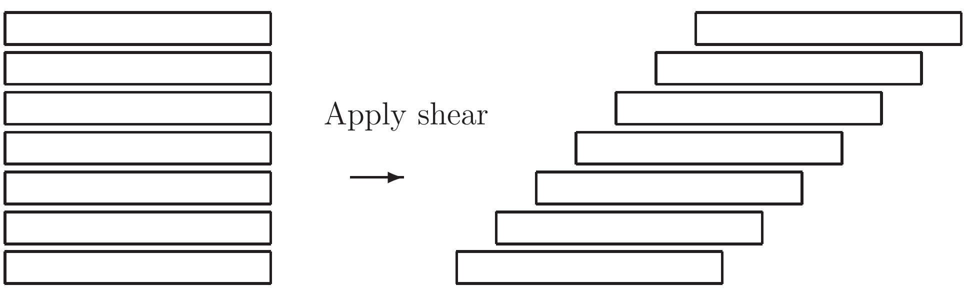
$$x = x_0 \sin(\omega_r t) e^{-\omega_0 t/2Q}\,, \qquad \omega_r = \omega_0\sqrt{1 - 1/(4Q^2)}\,. \tag{7.42}$$

Chapter 8

Viscosity

8.1 Shear flow in fluids

Physicists use the term *shear* to describe a deformation where different "sheets" of a surface slide past each other:



A *solid* is anything which will bounce back after being sheared. When you deform the solid, it will resist and will continue to try to spring back until you finally quit applying the forces which keep the material sheared. Then it will spring back to its original shape. (Unless it breaks, of course.)

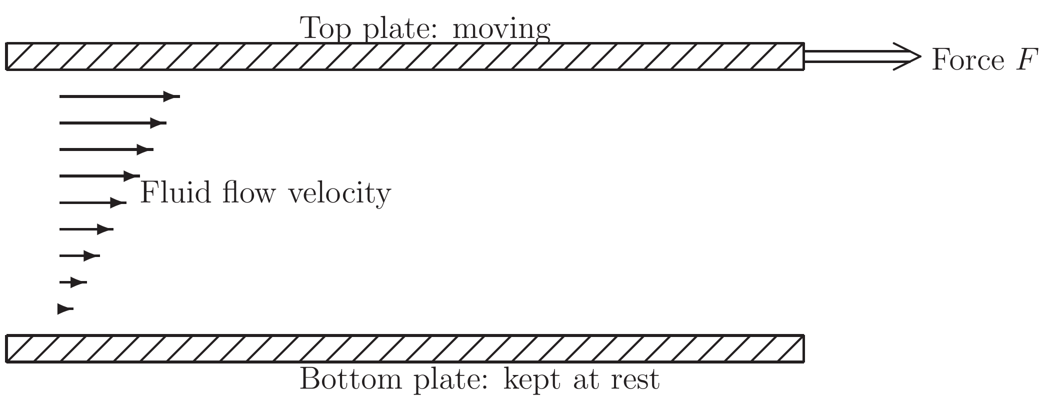
A *fluid* is anything which, when sheared, will remain in the new sheared state without any internal strain or tendency to "bounce back." Pour a fluid from one container to another and it takes the shape of the new container without any tendency to bounce back to the shape of the original container. However this does not mean that a fluid shows no resistance to the process of shearing. In an analogy to the spring-mass-damper resonator we discussed in the last section, a fluid has a zero value of K the spring constant–if you push it into a new shape it shows no springyness to go back to the old shape. However it can have a nonzero value of R the friction–it can resist the actual *motion* it takes to change it from one shape to another.

The reason we care is that the velocity of any fluid, such as air, right at the surface of a solid is generally zero. The way to understand this is to think about the air molecules which bounce off the surface. The surface does *not* act like a “mirror” for air molecules. It is composed out of atoms or molecules of the solid which are in constant vibration and motion. When an air molecule hits a solid molecule they exchange motional energy and the air molecule comes away with a random velocity and direction. If the air is moving across the surface, then each time an air molecule hits the surface it loses this “across” motion and replaces it with a random motion. This drags the velocity of the air right at the surface of the solid down to zero.

However if you look at the solutions we found for sound inside of a tube, we found that the air was moving *along* the tube, and that the velocity was the same across the whole cross-section of the tube. We have made another approximation–failing to take into account that the motion has to stop right at the edge of the tube. But if that is the case, then the air in the tube is under shear: the air in the middle of the tube is moving forward, the air right against the walls is at rest. Therefore we have to understand and include the shear resistance presented by the fluid, which will dissipate some of the energy of the sound wave.

8.2 Shear viscosity

The resistance of a fluid to shear stress is described by its *shear viscosity*. The defining experiment for shear viscosity is the following. Trap a fluid between two plates of area A and separation d. Keep the bottom plate at rest and pull the top plate:



To keep the plate moving at a steady velocity v will require a force F which is related to the other parameters as follows. The bigger the area, the bigger the force. The larger a forward velocity, the bigger a force. The larger the plate separation, the *smaller* the force–after all, the amount of “shearing” of the fluid is less in this case. Therefore

$$F = \frac{vA}{d} \times \eta \,. \tag{8.1}$$

Here η is some property of the fluid, called the *shear viscosity*, which serves as the constant of proportionality in the above. In other words, the viscosity tells how much a fluid resists shear flow; shear flow is v/d the change in velocity with distance, and the viscosity tells the force per area (same units as pressure) required to maintain a certain shear flow v/d. The units of viscosity are Newton-seconds per meter squared, N s/m^2 or Pascal-seconds Pa s, or kg/m s. The viscosities of our two favorite substances are:

- **Air**: 1.82×10^{-5} kg/m s at 20° Celsius. The viscosity of air is *independent* of pressure but rises slowly as the temperature is increased.

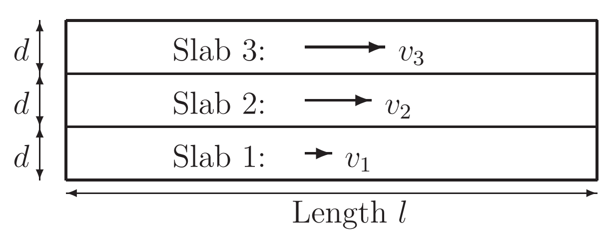
- **Water**: 1.0×10^{-3} kg/m : s at 20° Celsius. The viscosity of water diminishes with rising temperature.

Physically, the viscosity of air comes about because the molecules fly freely between scatterings. If the molecules flew freely all the way from one plate to the other, then molecules flying up to the top plate would not be moving forward (since they came from the bottom plate which is at rest). They would pick up forward motion when they bounced off the top plate, pushing it backwards according to Newton's third law. Then they would carry that forward motion and impart it to the bottom plate. Since individual molecules actually fly only a short distance between scatterings, the force they transfer is much less than it would be in this "free particle" limit.

8.3 Viscosity in pipes

Here is a practical application of viscosity. Consider the steady flow of a fluid (like water) inside a pipe (like a plumbing pipe). You can guess that viscosity will impede the flow through the pipe, so the pressure has to be higher on one end of the pipe than on the other to keep the fluid flowing. Therefore the thing driving the water is the gradient of pressure (how fast the pressure falls away with length along the pipe) and the thing slowing the water is the viscosity; the water velocity will take the value which balances these effects.

To see how it works, think about three slabs of fluid. Each slab is d tall, w wide and l long. Call the velocities of three slabs v_1, v_2, and v_3.



Think about the middle slab. There are four forces acting on it:

- the pressure forces on the front and on the back. If the pressure difference is ΔP, the total force is $wd\Delta P$.

- the viscous drag from the fluid below our slab. The drag is the area times the velocity difference per unit height times the viscosity,

$$F = -wl\frac{v_2 - v_1}{d}\eta\,. \tag{8.2}$$

- the viscous drag (or, if $v_3 > v_2$, viscous forwards pull) from the fluid above the slab.

$$F = vl\frac{v_3 - v_2}{d}\eta\,. \tag{8.3}$$

For the fluid to flow at a steady speed, these forces have to balance. (When you first turn on the water, the force from pressure speeds up the water for a moment. When you turn off the water, the overpressure from its slamming into the shut valve slows it down. In between, the water flows at a steady speed set by balancing these forces.) Therefore

$$wd\Delta P = wl\eta\left(\frac{v_2 - v_1}{d} - \frac{v_3 - v_2}{d}\right)\,. \tag{8.4}$$

Some of you will recognize $\frac{v_2 - v_1}{d} = \frac{dv}{dx}$ as the derivative of the flow with height. The two derivatives we need are the value at the top of the slab and the value at the bottom of the slab–points separated by a distance d. Multiply and divide by d:

$$\begin{aligned} wd\Delta P &= wl\eta\left(\frac{v_2 - v_1}{d} - \frac{v_3 - v_2}{d}\right) \\ &= wl\eta\left(\frac{dv}{dx}[\text{bottom}] - \frac{dv}{dx}[\text{top}]\right) \\ &= wdl\eta\left(\frac{\frac{dv}{dx}(x) - \frac{dv}{dx}(x+d)}{d}\right)\,. \end{aligned} \tag{8.5}$$

But for the same reason, this difference of first derivatives separated by d is just the second derivative (actually, minus the second derivative). Also, we can divide by wdl to find:

$$\frac{\Delta P}{l} = -\eta\frac{d^2 v}{dx^2} = -\eta\nabla^2 v\,. \tag{8.6}$$

The version with ∇^2 is the multi-dimensional generalization. It just means $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}$ (if z is the direction v is traveling and it is uniform in that direction).

Let us apply this to a circular pipe. As usual, the details of the derivation are for the interested, the answer is the important part. The water velocity at the wall of the pipe $r = a$ will be zero; I need to solve for it inside the pipe, $r < a$. Note that

$$\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right] r^2 = \left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right](x^2 + y^2) = 4 \tag{8.7}$$

therefore $r^2/4$ has $\nabla^2(r^2/4) = 1$ is a function with constant second derivative. A function with constant second derivative which vanishes at the wall of the pipe is $(a^2 - r^2)/4$, with $\nabla^2(a^2 - r^2)/4 = -1$. Therefore the solution we want is

$$v(r) = \frac{a^2 - r^2}{4\eta} \frac{\Delta P}{l}\,. \tag{8.8}$$

The flow rate is fastest in the middle of the pipe and falls to zero at the walls. As you make a pipe bigger, the flow speed in the middle of the pipe rises as the square of the pipe radius.

The peak flow velocity (at the centre of the pipe) is $(a^2/4\eta)(\Delta P/l)$. The average flow speed smaller; at the edges of the pipe the flow speed is zero. The average turns out to be half the peak value, so the total flow is πa^2 (the area) times half the peak flow velocity. Alternatively, if you can integrate, you may perform the integral of Eq. (8.8) over the cross-section of the pipe. The answer is

$$\text{flow} = 2\pi \int_0^a r dr v(r) = \frac{\pi a^4 \Delta P}{8\eta l}\,, \tag{8.9}$$

which is the same as $\pi a^2 v_{\rm peak}/2$, as I said.

Unsurprisingly, the flow rate of the water doubles if you double the pressure pushing the water. Also unsurprisingly, long pipes allow less flow (or take more pressure to keep it moving). We are also not surprised that viscous fluids (η large) allow little flow, while low viscosity fluids allow much more flow. More surprisingly, the flow goes as the *fourth* power of pipe radius; a pipe of 2 times the radius carries 16 times as much fluid!

8.4 Sound in a pipe

What about a sound wave in a pipe? A key assumption in the treatment of flow in pipes above was that the fluid is in a steady state. But when there is a sound in a pipe, the flow direction switches back and forth at the frequency of the sound, typically hundreds of times a second. Therefore that approximation will be no good; we have to include the fact that the air is speeding up and slowing down. Now Eq. (8.5) is just force balance–that the total force on a "box" of fluid should be zero. Instead, use Newton's equation to set the total force equal to ma the mass times acceleration:

$$-wd\Delta P - wld\,\eta \frac{d^2 v}{dx^2} = ma = (wld\rho)\frac{dv}{dt}\,. \tag{8.10}$$

Divide by the volume:

$$\frac{dP}{dz} = \eta \frac{d^2 v}{dx^2} - \rho \frac{dv}{dt} \qquad \text{or} \qquad \boxed{\frac{dP}{dz} = \eta \nabla^2 v_z - \rho \frac{dv}{dt}}\,. \tag{8.11}$$

Again, the version with ∇^2 is the result when v varies in several directions.

In the middle of a pipe the solution for a forward moving wave will be the usual one: $v = P/Z$. At the boundary the velocity will vanish. Near the boundary, part of the pressure dP/dz is getting used to make the fluid move back and forward, and part is used "shearing" the fluid. Just off the wall, the pressure gradient is mostly shearing the fluid; but as you go inwards and the velocity of the fluid increases, more and more of dP/dz is keeping the fluid moving and less and less is used up in shear. But it is not obvious just thinking about it, exactly how the velocity will rise up to the "normal" value near the middle of the pipe.

So let us calculate the actual velocity profile in the pipe. Do not worry if you don't follow the details below, which are advanced. What we need is the answer, or at least the general form of the answer.

Consider a forward-moving wave in a pipe. The pressure is $P = P_0 \cos(\omega t - 2\pi z/\lambda)$. We will use the complex notation tricks we just learned to rewrite this as $P = \Re\, P_0 e^{i(\omega t - 2\pi z/\lambda)}$. Now we have to guess how the velocity will behave. The complicated behavior will be very close to the wall, so we will approximate that the wall is planar and define x as the distance from the wall. Our guess for the velocity is that it rises towards the "normal" value P/Z exponentially:

$$v = \frac{P}{Z}\left[1 - e^{-x/x_0}\right] . \tag{8.12}$$

Sticking this into Eq. (8.11) we find

$$-i\frac{2\pi P_0}{\lambda} = -i\omega\rho\frac{P_0}{Z}\left[1 - e^{-x/x_0}\right] + \eta\frac{P}{Z}\left[-\frac{1}{x_0^2}e^{-x/x_0}\right] . \tag{8.13}$$

Now use that $Z = \rho c_{\mathrm{s}}$ and $c_{\mathrm{s}} = f\lambda = 2\pi\omega\lambda$. That means that $-i\omega\rho P_0/Z = -i\frac{2\pi P_0}{\lambda}$. So the 1 terms cancel. That leaves

$$0 = i\omega\rho\frac{P_0}{Z}e^{-x/x_0} - \frac{\eta}{x_0^2}\frac{P}{Z}e^{-x/x_0} . \tag{8.14}$$

These terms will cancel provided that

$$x_0^2 = i\frac{\eta}{\omega\rho} , \qquad \rightarrow \qquad x_0 = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\eta}{\omega\rho}} . \tag{8.15}$$

That means that the boundary layer does look like $1 - e^{-x/x_0}$ but with x_0 complex.

What is important (you can start listening again if you are equation phobic) is that there is a thin layer where the velocity goes from the usual value down to 0 as it must on the wall of the tube. The width of this boundary layer is roughly

$$\sqrt{\frac{\eta}{\omega\rho}} . \tag{8.16}$$

You can check that the units work correctly:

$$\sqrt{\frac{\mathrm{kg}}{\mathrm{ms}}\frac{1}{\mathrm{s}^{-1}}\frac{\mathrm{m}^3}{\mathrm{kg}}} = \sqrt{\frac{\mathrm{kg\ s\ m}^3}{\mathrm{ms\ kg}}} = \sqrt{\mathrm{m}^2} = \mathrm{m} \tag{8.17}$$

The boundary layer is thin for low viscosity and high frequency, but only as a square root. Typical values might be:

$$\sqrt{\frac{\eta}{\omega\rho}} \text{ at 440 Hz } = \sqrt{\frac{1.8 \times 10^{-5}\text{kg/ms}}{2\pi \times 440/\text{s} \times 1.2\text{kg/m}^3}} = 0.000074 \text{ m} = 74\ \mu\text{m}\,, \tag{8.18}$$

less than 1/10 a millimeter or a little thicker than a human hair. The boundary layer is thicker for lower frequency sounds and thinner for higher frequency sounds. Under normal conditions the values of η and ρ are basically always the same.

The thickness of the boundary layer also tells us how smooth the walls of a tube need to be. Smoothness is a measure of the heights of bumps or unevennesses. Provided that any bumpiness is shorter than this boundary thickness, it will have little effect on how air flows. Surfaces with bumpiness larger than 50 microns will feel rough to the touch; so generally if a surface feels smooth to the touch, it is smooth so far as sound is concerned.

Next we have to figure out how much energy gets lost because of viscosity. Power is force times velocity. The power lost to viscosity is the force due to viscosity dotted with the velocity. The force per volume from viscosity is $\eta\nabla^2 v$, as we saw. That means that we need to sum $\eta v\nabla^2 v$ over the volume of the instrument. The power absorbed in this way is lost to the sound wave and becomes heat in the air.

I will skip the awful details. The answer which comes out at the end is that

$$\frac{dE}{dt} = E \times \left(\omega \times \frac{\sqrt{2}}{a}\sqrt{\frac{\eta}{\rho\omega}} \right) . \tag{8.19}$$

The energy loss is as usual proportional to the energy in the system. We can read off the resonant Q:

$$\frac{dE}{dt} = \frac{\omega E}{Q} = E\omega\frac{\sqrt{2}}{a}\sqrt{\frac{\eta}{\rho\omega}} \qquad \rightarrow \qquad Q = \frac{a}{\sqrt{2}}\sqrt{\frac{\rho\omega}{\eta}} . \tag{8.20}$$

Note that the cross-sectional area of the pipe is πa^2; the part of this which in the boundary layer is the perimeter $2\pi a$ times thickness of the boundary layer $\sqrt{\eta/\rho\omega}$. Therefore the fraction of the pipe's cross-section made up by the boundary layer is $(2/a)\sqrt{\eta/\rho\omega}$. This is very close to $1/Q$; it differs only by a factor of $\sqrt{2}$, which arises because of the funny complex exponential form of the exponential tail. Physically we can understand this form for Q as follows: the sound in the boundary layer is getting "eaten" by viscosity while the sound in the rest of the instrument is oscillating. After one oscillation the sound in the rest of the instrument has to "refill" the sound energy in the boundary layer; hence the $1/Q$ (fraction of energy lost per $1/\omega$ time) should be given by the volume fraction in the boundary layer.

For reasonable numbers like $a = 0.01$ m (1 cm) and $f = 440\,\text{Hz}$, the value of Q we find is $Q = (0.01\,\text{m}/(2 \times .000074\,\text{m})) = 95$. That gives a decay time of $95/(2\pi \times 440\,\text{Hz}) = .034\,\text{s}$. For comparison, in moderato tempo, a quarter note is 0.5 seconds and a 32'nd note is .063 seconds.

This treatment also gives us a better idea of how cloth damps sound. Every fiber has a boundary layer $x_0 \sim 10^{-4}$ m thick around it where the sound energy gets absorbed by friction. If something is "fuzzy" then its effective area could be very large.

8.5 Thermal conductivity: total Q

There is another way that the walls of the tube cause energy dissipation: thermal conductivity. We saw when we discussed how sound works that the high pressure regions are also hotter, and the low pressure regions are also colder. That means that, as a sound wave propagates down a tube, it is causing the air against any spot in the tube to be alternately warmer and cooler. When it is warmer, heat moves out of the air into the wall of the instrument. The air is therefore cooler than it should be, and the pressure does not rise as much as it should. Similarly, in a rarefaction, the air cools and absorbs heat from the walls. This raises the pressure so that it is not as low as it should have been. This weakens the sound wave; sound is dissipated into a net heating of the walls and air.

The analysis is similar to the one for shear viscosity. Heat does not diffuse very quickly through air, so again there is only a boundary layer which exchanges heat with the walls of the instrument. The boundary layer is somewhat thicker than the viscous boundary layer; but the heating and cooling of the air in a sound wave is a subdominant effect, suppressed by $(\gamma - 1) \simeq 0.4$. Skipping the details, the punchline is that the dissipation from this mechanism is about 1/2 as large as the dissipation by viscosity. Therefore it is a good approximation just to multiply the amount of energy dissipated by viscosity, found above, by 1.5 to account for thermal conductivity effects. That is the same as multiplying the Q due to viscosity by $1/1.5 = 2/3$ to incorporate both loss mechanisms.

If energy is lost in multiple ways, what is the Q of the resonator? Do the values of Q from different loss mechanisms add to give the total Q? **NO.**

Recall the definition:

$$Q = \frac{\omega E}{-dE/dt}\,, \qquad \text{or} \qquad -\frac{dE}{dt} = \omega E \times \frac{1}{Q}\,. \tag{8.21}$$

Now suppose energy gets lost by 3 mechanisms (such as sound radiation, viscosity, and thermal conductivity), which would each separately have values Q_1, Q_2, and Q_3. The total energy loss rate is

$$-\frac{dE}{dt} = \omega E \frac{1}{Q_1} + \omega E \frac{1}{Q_2} + \omega E \frac{1}{Q_3} = \omega E \frac{1}{Q_{\rm tot}}\,. \tag{8.22}$$

We see therefore that

$$\frac{1}{Q_{\rm tot}} = \frac{1}{Q_1} + \frac{1}{Q_2} + \frac{1}{Q_3}\,. \tag{8.23}$$

One adds *inverses* of Q's to get the *inverse* of the total Q. That means that the Q of a resonator will be lower than the lowest value from any of the dissipation mechanisms.

We have not figured out the contribution to Q from radiation (yet). The rate of energy loss is $dE/dt = (2\pi a/\lambda)^2$ times the power incident on an opening. The time it takes for sound to go the length of the instrument and back is $2L/c_\mathrm{s}$. Therefore, every $2L/c_\mathrm{s}$ of time, all of the energy bounces off the opening once. Therefore

$$\frac{dE}{dt}[\text{incident}] = \frac{E}{2L/c_\mathrm{s}}\,. \tag{8.24}$$

Therefore

$$\frac{dE}{dt}[\text{radiated}] = \frac{4\pi^2 a^2}{\lambda^2}\frac{Ec_\mathrm{s}}{2L} = \frac{\omega E}{Q} \qquad \rightarrow \qquad Q_\text{radiation} = \frac{\lambda^2}{4\pi^2 a^2}\frac{2L\omega}{c_\mathrm{s}}\,. \tag{8.25}$$

Now use that $\omega = 2\pi f = 2\pi c_\mathrm{s}/\lambda$:

$$Q_\text{radiation} = \frac{\lambda^2}{4\pi^2 a^2}\frac{2L2\pi c_\mathrm{s}/\lambda}{c_\mathrm{s}} = \frac{\lambda L}{\pi a^2}\,. \tag{8.26}$$

This is *per* opening of the instrument.

Totalling up the sources of dissipation found so far, we find

$$\frac{1}{Q} \simeq \frac{1}{a}\sqrt{\frac{2\eta}{\rho\omega}} \times \left(1 + \frac{1}{2}\right) + \sum_\text{openings} \frac{\pi a^2}{\lambda L}\,. \tag{8.27}$$

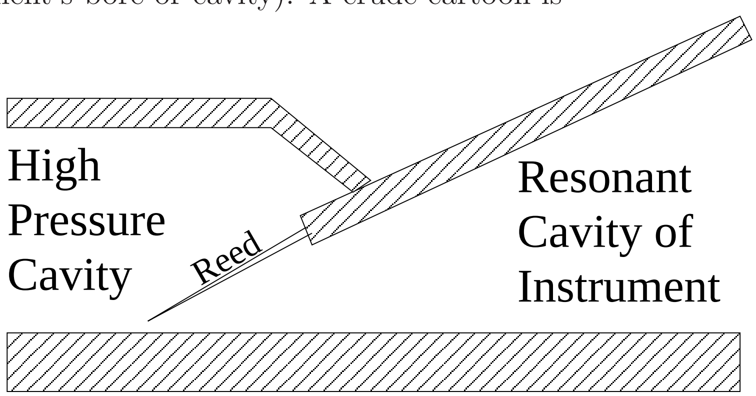
The first "viscous" term gets larger at low frequencies; the second "radiation" term gets larger at high frequencies (since $f \propto 1/\lambda$).

Chapter 9

Reeds

This chapter will be very brief because the material is also covered in other notes I have given you. For a survey of the idea of a reed see the chapter on reeds in the Phys 224 notes.

A reed is a pressure controlled valve. That means it is a narrowing in an airstream (or a divider between two chambers which has a small opening for air to go through), and that the size of the "gap" for air to go through varies depending on the pressures in the two chambers. In the following I will call the two chambers "the mouth" and "the cavity" (meaning the end of the instrument's bore or cavity). A crude cartoon is



The picture illustrates a clarinet reed, which is a thin piece of cane or bamboo, tapered near the tip. It is flexible but with rigidity. If you bent the reed down and released it, it would vibrate back and forth as a (damped) resonator. The other key aspect is that the reed has surfaces which are in contact with both air cavities. The pressures in those cavities push on the reed and bend it. For the reed shown (an inward-opening reed), if the mouth cavity is high that bends the reed down, towards closing off the air channel. Pressure in the cavity bends the reed up, opening the channel. If the reed bends down far enough, its tip touches the bottom plate of the instrument and the air channel is closed off.

For a clarinet reed, the reed leaves open an air channel which has a fixed width w but a height which depends on the reed position–namely, on the distance between the tip of the reed and the bottom of the mouthpiece. Call that distance x. Then the opening area is

$S = wx$.

As mentioned, the reed acts like a spring-mass system. The position x is controlled by three things: inertia, the springiness of the reed, and the forces on the reed because of the pressures in the two cavities. We expect the reed opening to obey an equation like

$$m_{\rm eff}\frac{d^2x}{dt^2} = F = -k(x-x_0) - (P_{\rm mouth} - P_{\rm cavity})A_{\rm eff} \tag{9.1}$$

where $m_{\rm eff}$ is the mass of the reed–correcting for the fact that the tip moves more than anywhere else on the reed, so it is really less than the full mass of the reed. The spring constant k tells how much force the stiffness of the reed exerts to resist the reed's bending, and x_0 is the size of the gap x when the reed is "at rest" with no forces acting on it. And $A_{\rm eff}$ is the "effective area" over which the pressures push. $A_{\rm eff}$ is smaller than the full area of the reed for the same reason that $m_{\rm eff}$ is smaller than the mass of the reed–the tip of the reed moves more than anywhere else. In general there should also be a damping term: many reeds [clarinet, oboe, bassoon, saxophone] are held against the lip, which is a soft, damping structure; and other reeds [brass instruments] actually *are* the lips.

There is no simple way to calculate what $m_{\rm eff}$, k, or $A_{\rm eff}$, though they are measurable for a specific reed system. x_0 can be set by the placement of the reed and is varied during play by applying force with the lip holding the reed.

We already know that the equation for x allows for sine wave oscillations (which will be damped if we include a resistance term). The natural oscillation frequency is $\omega_0 = \sqrt{k/m_{\rm eff}}$. For some instruments [woodwind reed instruments] this frequency is much higher than the notes in the normal register of the instrument. In that case, based on what we learned in Chapter 7, we can ignore the mass term and solve for x just from force balance:

$$0 = -k(x-x_0) - (P_{\rm mouth} - P_{\rm cavity})A_{\rm eff} \qquad \rightarrow \qquad x = x_0 - \frac{A_{\rm eff}}{k}(P_{\rm mouth} - P_{\rm cavity})\,. \tag{9.2}$$

From now on I will define

$$P_{\rm mouth} - P_{\rm cavity} \equiv \Delta P$$

so what we learned is that this pressure difference causes the gap in the reed to close up.

What we really care about is how fast air flows through the reed. That is because the power imparted into the sound wave inside the pipe is Power = force×velocity = $(P_{\rm cavity} - P_{\rm atmos}) \times U$ with U the airflow. So we want to know how U depends on $P_{\rm cavity}$. If a higher $P_{\rm cavity}$ causes more airflow, then the airflow will be adding energy to the sound wave in the instrument.

We need to know how fast air moves through the gap under the reed. To answer that, we need to know about the Bernoulli effect. Basically, it takes a pressure drop to get air moving. The long-winded derivation way to see this is to start with Eq. (8.11)

$$\frac{dP}{dz} = \eta\nabla^2 v - \rho\frac{dv}{dt}\,. \tag{9.3}$$

We can check after the fact that the viscosity term here will not be important for our problem. Drop it. Then, multiply the equation by $v = dz/dt$ (which I am free to do):

$$\frac{dz}{dt}\frac{dP}{dz} = -\rho v \frac{dv}{dt} = -\frac{\rho}{2}\frac{dv^2}{dt} \,. \tag{9.4}$$

Use the chain rule: $(dz/dt)(dP/dz) = (dP/dt)$. Therefore

$$\frac{d}{dt}P = \frac{d}{dt}(-\rho v^2/2) \,. \tag{9.5}$$

This equation says that, as air moves from high to low pressure, the pressure change equals $-\rho/2$ times the change in the velocity squared. If the air in the mouth is not really moving, we find that

$$\frac{\rho v^2}{2} = \Delta P \qquad \rightarrow \qquad \boxed{v = \sqrt{\frac{2\Delta P}{\rho}}} \,. \tag{9.6}$$

If we put in reasonable numbers for a reed, like $\Delta P = 1000$ Pa (which is 1/100 of an atmosphere or 1/10 of your blood pressure), we find $v \simeq 41$ m/s. The maximum pressure you can exert is around 12 000 Pa (enough to turn your face red), which gives $v \simeq 140$ m/s [which is 500 km/hour–yes, when you purse your lips very tight and blow through them very hard, the air between your lips is moving at hundreds of km/hour].

Now we check that viscosity did not matter. If the gap between the reed and the mouthpiece is 1 mm then $\nabla^2 v \sim v/(1 \text{ mm})^2$ which is something like 10^9/m s. Multiply by $\eta \sim 10^{-5}$kg/m s and you get 10^4 kg/m^{2}s^2, which sounds big. But dP/dx is 1000 Pascal over 1 mm is 10^6 kg/m^{2}s^2, much much bigger. Therefore viscosity is negligible.

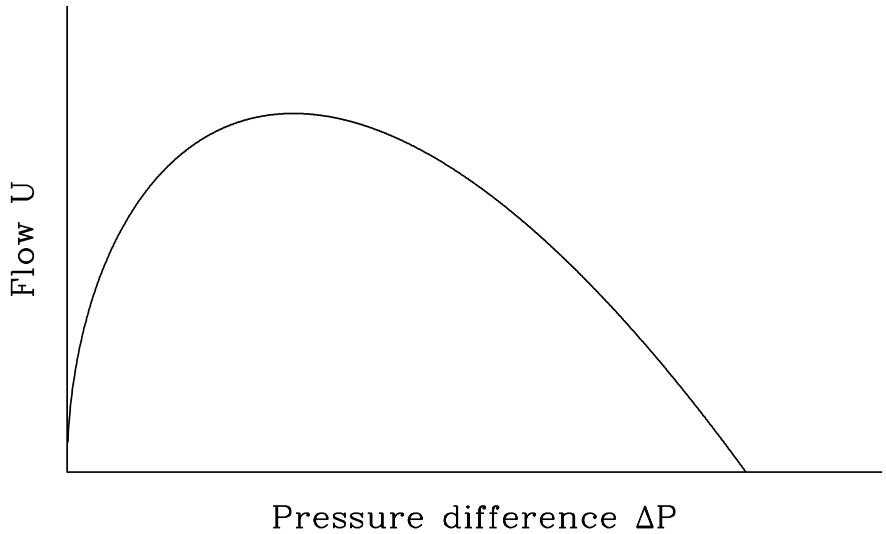
But what we wanted was the total airflow. This is v times the area wx:

$$U = vS = \sqrt{\frac{2\Delta P}{\rho}}\, w \left(x_0 - \frac{A_{\rm eff}}{k}\Delta P \right) \,. \tag{9.7}$$

What I care about right now is that this can be written

$$U = \left[w x_0 \sqrt{\frac{2}{\rho}} \right] \Delta P^{1/2} - \left[\frac{w A_{\rm eff}}{k} \sqrt{\frac{2}{\rho}} \right] \Delta P^{3/2} = C_1 \Delta P^{1/2} - C_2 \Delta P^{3/2} \,. \tag{9.8}$$

Here C_1 and C_2 are some numbers which depend on the details of the reed; presumably you can change the shape and form of the reed to adjust what they are. The important part is the functional form:



As the pressure difference is raised, the air flow at first increases (naturally–more pressure is pushing the air through the reed). Then it tops out and decreases (naturally–the pressure difference is forcing the reed shut).

Is the change in airflow U with pressure ΔP enough to get the cavity to resonate: That depends. Considering changing ΔP by some amount δP. Call the change in flow δU. Then $\delta U/\delta P$ is how much the flow increases per amount of pressure change. This is an *admittance*. If it is bigger than the emittance of the cavity (if a sound wave in the cavity has less than this amount of flow per amount of pressure change), then the extra flow represents a strengthening of the sound wave in the cavity.

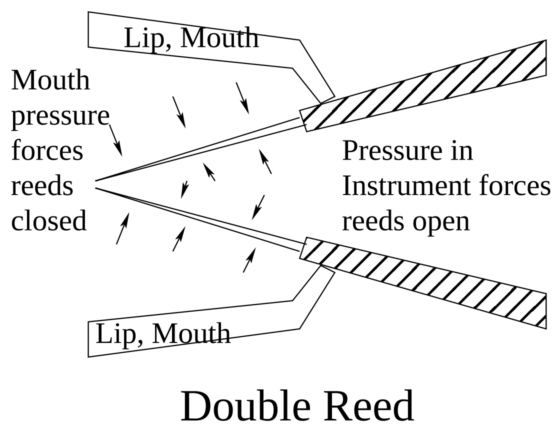
That means that one must determine the admittance of the cavity: if it is smaller than this "reed admittance," then the reed can amplify vibrations. Alternately, the cavity *impedance* must be larger than $\delta P/\delta U$ the reed impedance. That means that we will have to study impedances of cavities.

There are two important variations on the reed just described: double reeds and lip reeds.

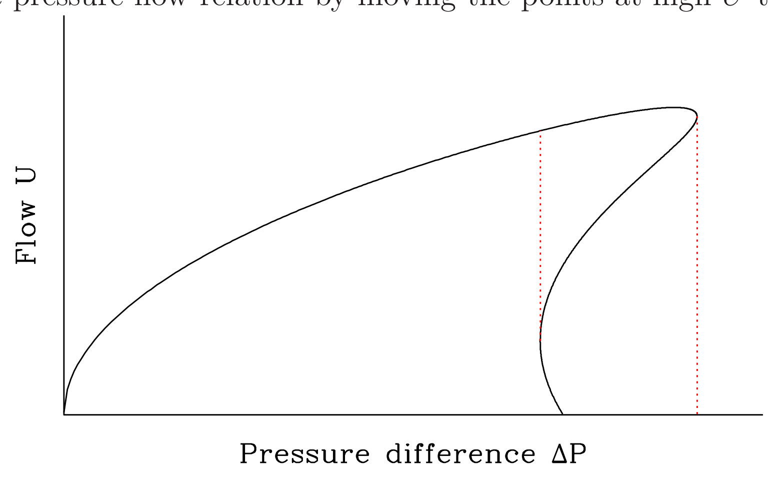
A "lip reed" is what you are using your lips to do when you buzz them (either when playing a brass instrument of when being very rude). The advantage of the lip reed is that you have direct muscular control over the features of the reed: how open or shut it is, how stiff it is (k), the effective mass ($m_{\rm eff}$). The "problem" is that an increased mouth pressure forces the lips open, instead of forcing them closed. That is, $A_{\rm eff}$ in Eq. (9.1) above is negative. If you follow through the rest of the argument about reeds, you find that U strictly increases as you raise ΔP, and so there is no case for which the sound in the resonant chamber spontaneously gets louder.

Of course, the discussion for clarinet type reeds assumed that the resonant frequency in the cavity was much lower than the resonant frequency of the reed. For lip reeds, this will not work. But if we look at the behavior of a resonator like Eq. (9.1) at higher frequencies, we see that for $\omega > \omega_0$, the position x is almost 180° out of phase with the driving force, see Figure 7.2. That means that, for frequencies $\omega > \omega_0$, the lips are most closed when ΔP is highest and are most open when ΔP is lower. We will get an increased flow for increased cavity pressure (lower ΔP). But it tells us that the instrument can only play notes with frequencies higher than the resonant frequency of the lips. We also see in Figure 7.2 that the response is biggest for frequencies very close to ω_0. That means that the lips will open and close the most for a frequency close to ω_0. Therefore, if an instrument has many resonant frequencies, the frequency which will play is the one with $\omega > \omega_0$ but by the smallest amount. In other words, by tuning the tension of your lips, you can make the instrument hop from resonant frequency to resonant frequency–a major part of how brass players control what note they play.

The other case to consider is the double reed. In a double reed, there are two pieces of cane which close towards each other. Naively this behaves just the same as a single reed: the mouth pressure forces the reed closed and the cavity pressure forces the reed open. The difference is that the width of the cavity (not shown here) is wide where the reed height is short and is narrow where the reed height is tall. That means that the channel which the reed opens into has about the same area as the slit through the reed (when the reed is open).



As a result, the air coming out from the reed travels some distance in a very narrow channel (called the bocal). The viscous resistance in this channel *is* important, and it results in a "back-pressure": the pressure on the inside of the reed is $P_{\text{cavity}} + P_{\text{backpressure}}$. The back-pressure, which is due to the airflow through the bocal, depends on the flow velocity! This modifies the pressure-flow relation by moving the points at high U towards high ΔP:



As ΔP is increased from zero, the flow steadily increases. This flow creates a back-pressure which helps keep the reed open. But at the "tip" in the figure, ΔP is so large that the flow starts to fall. As it does, the back-pressure disappears and the reed suddenly falls shut, following the (red) dotted curve. If ΔP is reduced, the reed remains shut for some range of pressures. When finally it starts to open again, there is a point when the rising current starts generating a significant back-pressure and the reed pops open, shown by the left dotted red curve.

The pressure cycle in a double reed therefore involves abrupt openings and closings of the reed. In the pressure range between the dotted line, there are two stable configurations; either the reed can be open, maintained by the back-pressure, or it can be closed, allowing no back-pressure. These sharp features in the reed's motion lead to a tone extremely rich in harmonics (very high frequency content), which is a characteristic of double reeds' tone quality (timbre).

Chapter 10

Input Impedance

In a reed instrument–or other wind instrument–there is one spot where the sound is being produced. The rest of the instrument acts as a resonant chamber, amplifying the sound and giving feedback to the sound producing element (such as a reed). In the last chapter we saw that, from the point of view of understanding the reed's behavior, what we need to know is the ratio of pressure to flow for a resonance in the instrument's body, when measured where the instrument's body meets the reed. That is what we will figure out how to compute in this lecture. The name for the ratio of a pressure to a flow is "impedance" as we have already seen. The actual size of pressure or of flow depends on how loud a sound is present; but for a fixed frequency of sound, the ratio is a property of the instrument. That is, given the frequency of a sound and the pressure (which determines the loudness), physics determines what the flow must be.

The philosophy of the calculation is that there is a sound source at the location of the reed. Because of it, there will be sound everywhere in the instrument. At the far end of the instrument (which is generally an open end), we know that sound is only emitted. That allows us to figure out the P/U ratio at that point. This will let us figure out the ratio of forward to reflected sound (including the relative phase) at other points in the instrument; we work our way back to the reed, to figure out the ratio P/U which the reed needs to produce. The whole calculation is done assuming the reed makes a steady tone at one frequency ω. [Knowing this behavior for each ω, we can figure out from the physics of the reed, what frequency or frequencies the reed will actually produce.]

10.1 Intrinsic impedance

The first concept is the "intrinsic impedance" of a tube. [All the impedances in this chapter are acoustic impedances. They are Pressure/Flow = Pascal / (volume/time) which has units of $\mathrm{kg/m^4\,s}$.] This is just the "impedance of a pipe" which we have already encountered. That is,

$$\boxed{\text{Intrinsic impedance: } \frac{P}{U} \text{ for one sound wave.}}$$

That is, it is the ratio of P/U if there is a forward moving wave and NO backward moving or reflected wave. We have already seen that its value is

$$\text{Intrinsic impedance:} \quad Z_0 = \frac{\rho c_{\rm s}}{S} = \frac{\rho c_{\rm s}}{\pi a^2} \, . \tag{10.1}$$

Here I write S for the area of the pipe, which is πa^2 with a the pipe radius (assuming the pipe has a circular cross-section, which is true of most pipes).

10.2 Load impedance

Impedance of an instrument's pipe is a function of location in the pipe, and generally does not equal Z_0. The reason is that there are generally two waves; a forward traveling wave, and a backwards traveling (reflected) wave. At an open end of a pipe, the pressures of forward and backwards waves cancel and the flows add, so the impedance at that point is small. Further back in the pipe, we are at a different point on both waves, so the answer will be different.

The first thing to figure out is the ratio of pressure to flow at the end of a pipe farthest from the reed. We will call the impedance at this point Z_L the "load impedance". For a closed pipe this is easy: the flow must be zero, and so

$$\boxed{\text{Closed end: } Z_L = \infty.}$$

Of course nothing is ever exactly infinity, but in this case it is pretty close; the only way there can be airflow is from the flexibility of the pipe material. So really Z_L is more like the mechanical impedance of the pipe material over S. This is typically thousands of times larger than Z_0 so taking it to be infinity is a good approximation.

For an open pipe, at the crudest approximation $Z_L = 0$, since the pressure must vanish. But we know that this is not exact; there are sound radiation and end corrections to account for. Consider radiation first. Suppose there is a sound wave moving towards the opening, and that the pressure associated with it, right at the opening, is P_f (f for "forward). There will also be a reflected wave. Call the pressure of the reflected wave P_b. The total pressure is $P_f + P_b$. We know that P_b and P_f are of opposite sign (the pressures nearly cancel at an opening), and for a narrow opening P_b nearly but not quite equal P_f. The ratio of intensities is

$$\frac{I_{\rm reflected}}{I_{\rm incident}} = \frac{P_b^2}{P_f^2} \, , \qquad \frac{I_{\rm radiated}}{I_{\rm incident}} = 1 - \frac{I_{\rm reflected}}{I_{\rm incident}} = \frac{P_f^2 - P_b^2}{P_f^2} \, . \tag{10.2}$$

But we also calculated

$$\frac{I_{\rm radiated}}{I_{\rm incident}} = \frac{4\pi^2 a^2}{\lambda^2} = k^2 a^2 \, . \tag{10.3}$$

Here I introduced $k = 2\pi/\lambda$ [called the "wave number"]. k is a convenient variable to use instead of λ for the same reason that we use ω instead of the frequency f: it absorbs factors of 2π. Equating these expressions,

$$k^2a^2 = \frac{P_f^2 - P_b^2}{P_f^2} = \frac{(P_f + P_b)(P_f - P_b)}{P_f^2}\,. \tag{10.4}$$

This gives us a relation between P_f and P_b.

What we want is $(P_f + P_b)/(U_f - U_b)$. The sign on U_b is reversed because the direction of air motion for a backwards moving wave is opposite of that for a forwards moving wave. We know $U_f = P_f/Z_0$ and $U_b = P_b/Z_0$. Therefore

$$Z_L = \frac{P}{U} = \frac{P_f + P_b}{U_f - U_b} = \frac{P_f + P_b}{P_f/Z_0 - P_b/Z_0} = Z_0\frac{P_f + P_b}{P_f - P_b} \tag{10.5}$$

Multiply top and bottom by $P_f - P_b$:

$$Z_L = Z_0\frac{(P_f + P_b)(P_f - P_b)}{(P_f - P_b)^2} = Z_0k^2a^2 \times \frac{P_f^2}{(P_f - P_b)^2} \simeq \frac{Z_0k^2a^2}{4}\,. \tag{10.6}$$

We also have to incorporate the end correction. Physically, the end correction means that P_f and P_b above are not quite in phase with each other. We know that the effect of the end correction is to make the tube effectively act $0.61\,a$ longer than it really is. For our purposes, the easiest way to deal with this is to just replace the tube length with the tube length plus $0.61\,a$. If you are unhappy with that you can incorporate it by adding $0.61\,ikaZ_0$ to Z_L (that is, a purely imaginary correction).

10.3 Propagation along the pipe

Next we need to figure out how to relate Z_L and Z_0 to the value of impedance anywhere else in the pipe.

There are two waves propagating in a pipe: the forward and backwards waves. If we know the load impedance Z_L, we can figure out the relation between their strengths at one point in the pipe. Then using Z_0 lets us figure out how they propagate in the pipe.

First let us understand how waves propagate in the pipe. Neglecting wall losses, we know the forward wave should vary as a sine wave–we will find it useful to use complex notation and write

$$P_{\rm forward}(x,t) = P_f e^{i(\omega t - kx)}\,, \qquad \text{with } \frac{\omega}{k} = c_{\rm s}\,. \tag{10.7}$$

The flow is related as

$$U_{\rm forward}(x,t) = P_{\rm forward}(x,t)Z_0^{-1}\,. \tag{10.8}$$

If we allow for the fact that there is damping on the walls, the sound wave should really decay. The decay with time is $e^{-\omega t/2Q}$. The factor of 2 is because Q tells how fast the energy decays with time: since energy goes as P^2, if P falls by $e^{-\omega t/2Q}$, then energy will fall as $P^2 \sim (e^{-\omega t/2Q})^2 = e^{-\omega t/Q}$. Here Q is the Q from Chapter 8 but including only viscous and thermal conductive effects–the parts that arose from friction on the walls of the pipe.

The situation we have in mind is that there is a steady sound source at the base of the pipe. The sound is always getting produced there. But as it propagates down the instrument's pipe, it loses loudness. Therefore the effect of the energy loss will be to make the sound decay with position; we really expect

$$P_{\rm forward}(x,t) = P_f e^{i(\omega t - kx)} e^{-kx/2Q}\,. \qquad (10.9)$$

If we define $k = |k|(1-i/2Q)$, then since $(-i)(-i) = -1$, $-ikx = -i|k|x - kx/2Q$. Therefore by using a complex k, we can include the effect of dissipation. In what follows we assume k is complex in this way.

Besides the forward wave there is of course a backwards wave of form

$$P_{\rm backward}(x,t) = P_b e^{i(\omega t + kx)}\,. \qquad (10.10)$$

The opposite sign ensures that the wave propagates backward; the imaginary part in k now means that the wave is largest at the far end of the instrument and decays towards the reed.

First consider a cylindrical pipe. Call the position at the far (load) end of the pipe $x = L$. Call the reed's position $x = 0$. What we know is that

$$\frac{P_{\rm forward}(L,t) + P_{\rm backward}(L,t)}{U_{\rm forward}(L,t) - U_{\rm backward}(L,t)} = Z_L\,. \qquad (10.11)$$

Using $U = P/Z_0$ and substituting our explicit formulae,

$$\frac{P_f e^{i(\omega t - kL)} + P_b e^{i(\omega t + kL)}}{\frac{P_f}{Z_0} e^{i(\omega t - kL)} - \frac{P_b}{Z_0} e^{i(\omega t + kL)}} = Z_L\,. \qquad (10.12)$$

Multiply top and bottom by $e^{-i(\omega t - kL)}$ and multiply top and bottom by Z_0:

$$Z_0 \frac{P_f + P_b e^{2ikL}}{P_f - P_b e^{2ikL}} = Z_L\,. \qquad (10.13)$$

Multiply through by the denominator:

$$Z_0 P_f + Z_0 P_b e^{2ikL} = Z_L P_f - Z_L P_b e^{2ikL} \qquad \rightarrow \qquad P_b = \frac{Z_L - Z_0}{Z_L + Z_0} P_f e^{-2ikL}\,. \qquad (10.14)$$

That tells us that the impedance anywhere else on the pipe is

$$\begin{aligned} Z(x) = \frac{P(x)}{U(x)} &= \frac{P_{\rm forward}(x,t) + P_{\rm backward}(x,t)}{U_{\rm forward}(x,t) - U_{\rm backward}(x,t)} \\ &= \frac{P_f e^{i(\omega t - kx)} + P_b e^{i(\omega t + kx)}}{\frac{P_f}{Z_0} e^{i(\omega t - kx)} - \frac{P_b}{Z_0} e^{i(\omega t + kx)}} \\ &= Z_0 \frac{P_f + P_b e^{2ikx}}{P_f - P_b e^{2ikx}} \end{aligned} \qquad (10.15)$$

which now applying Eq. (10.14), as well as

$$e^{iky} + e^{-iky} = 2\cos(ky)\,, \qquad e^{iky} - e^{-iky} = 2i\sin(ky) \tag{10.16}$$

becomes

$$Z(x) = Z_0 \frac{Z_L \cos[k(L-x)] + iZ_0 \sin[k(L-x)]}{iZ_L \sin[k(L-x)] + Z_0 \cos[k(L-x)]}\,. \tag{10.17}$$

Setting $x = 0$ gives the impedance at the reed.

The formula reproduces some things we already know. For an open-mouthed instrument, $Z_L \simeq 0$. Taking k real and $Z_L = 0$, the expression simplifies:

$$Z(x) = Z_0 i \tan[k(L-x)]\,, \qquad \text{(Open end, idealized).} \tag{10.18}$$

If the beginning $x = 0$ of the instrument is a closed end, there is a resonance at any frequency meeting the boundary condition $Z = \infty$. (A resonance means sound can exist without a source. Nonzero Z means that you have to generate a net flow at that point to keep a sound going.) This occurs for $\tan(kL) = \infty$, which occurs for $kL = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, or $L = \frac{\pi}{2k}, \frac{3\pi}{2k}, \ldots = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$. These are the wavelength to length relations we already found for resonances in open-closed tubes. On the other hand, if the tube is open at $x = 0$ then we need $Z = 0$ for a resonance to occur. This happens if $kL = n\pi$ with n an integer, which reproduces $L = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$.

Returning to a real instrument, if k is complex then the formula still applies; $\cos(kL) = \cos(|k|L - ikL/2Q)$. I need to know how to deal with complex arguments in sines and cosines. The answer is:

$$\cos(a+ib) = \cos(a)\cosh(b) - i\sin(a)\sinh(b)\,, \qquad \sin(a+ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)\,. \tag{10.19}$$

10.4 Varying-shaped pipes

What about pipes which vary in shape–which are the case in essentially all real instruments?

Now $Z_0 = \rho c_s/\pi a^2$ is a function of x:

$$Z_0(x) = \frac{\rho c_s}{\pi a^2(x)} \tag{10.20}$$

Our calculation of P_b in terms of P_f at the point $x = L$ is still valid. So is the calculation for the point x given the behavior at L, PROVIDED that the tube does not change shape between L and x. For very short separations this should be a good approximation; except for the final flare of a brass instrument (which cannot be treated as a pipe any more), the change in a with x is generally gradual and for short lengths you can take it to be uniform.

So pick $x = L - \delta$. Apply Eq. (10.17):

$$Z(L-\delta) = Z_0 \frac{Z(L)\cos(k\delta) + iZ_0 \sin(k\delta)}{iZ(L)\sin(k\delta) + Z_0\cos(k\delta)} \,. \tag{10.21}$$

Since I chose δ small, one can expand $\sin(k\delta) \simeq k\delta$ and $\cos(k\delta) \simeq 1$. Therefore

$$Z(L-\delta) = Z_0 \frac{Z(L) + iZ_0 k\delta}{iZ(L)k\delta + Z_0} \simeq Z_0(Z(L) + iZ_0 k\delta)\left(\frac{1}{Z_0} - \frac{ik\delta Z(L)}{Z_0^2}\right) \,. \tag{10.22}$$

Here I geometrically expanded the denominator to first order. Keeping only the terms to first order in $k\delta$,

$$Z(L-\delta) = Z(L) + ik\delta\left(Z_0 - \frac{Z_L^2}{Z_0}\right) \,. \tag{10.23}$$

Move $Z(L)$ to the lefthand side and divide by δ:

$$\frac{Z(L-\delta) - Z(L)}{\delta} = ik\left(Z_0 - \frac{Z(L)^2}{Z_0}\right) \,. \tag{10.24}$$

The lefthand side is a derivative. We can apply the same argument anywhere along the pipe. Therefore

$$-\frac{dZ(x)}{dx} = ik\left(Z_0 - \frac{Z^2(x)}{Z_0}\right) \,. \tag{10.25}$$

This is a *differential equation* which lets you solve for $Z(x)$ everywhere, given $Z(L)$ the initial value and $Z_0(x)$ the intrinsic impedance along the pipe. If you don't know how to solve an equation like this, do not worry. For complicated cases it has to be done numerically by a computer.

10.5 Finger holes

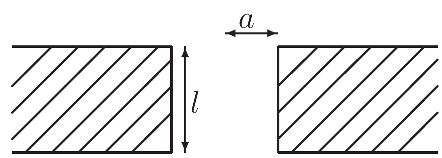
The other real-world complication for real wind instruments is that the woodwinds have finger holes. That means there are small holes cut in the bore of the instrument, which may be opened or closed (historically, by covering them with fingers; in modern instruments, by covering them with pads which are moved by levers controlled by fingers). Generally the finger holes are much smaller in diameter than the bore diameter of the instrument. It is not a good approximation to assume that the instrument simply ends at the location of the hole. Instead we should do an honest calculation of the behavior of the instrument bore with a hole in it.

To see how it works, think about a hole near the end of the instrument:

Call the length of instrument between the hole and the final opening of the instrument D. We can calculate the impedance the instrument *would* have, at the location of the hole, using formulae from Section 10.3. Call that impedance Z_D.

Now suppose there is a pressure P at the location of the hole. The flow which continues down the tube of the instrument is $U = P/Z_D$ (that is what Z_D means). But there will also be a flow out the hole.

To calculate it, we have to think more carefully about the hole. Call the area of the hole $S_{\rm hole}$ and its radius $a_{\rm hole}$ (so $S_{\rm hole} = \pi a_{\rm hole}^2$). The hole is like a tiny pipe in the side of the instrument, of length l equal to the thickness of the wall of the pipe.



The effective length of the pipe has an additional $2 \times 0.85\,a$ from the end corrections at its top and bottom (not 0.61 because they are flanged). In addition the instrument bore is sometimes thicker where the hole is, and there is a pad on top which funnels the air and may increase the end correction. The upshot is that the hole acts like a tube of nonzero effective length $l_{\rm eff}$.

Given a pressure P in the instrument bore at the location of the finger hole, what is the airflow? It is the *sum* of the air flow down the tube and the airflow out the hole. The airflow out the hole is $U_{\rm hole} = P/Z_{\rm hole}$; we need to compute the impedance at the base of the fingerhole. But we know how to do that. Taking Z_L of the tone hole to be zero (neglecting sound radiation from the hole), we would have

$$Z_{\rm hole} = i Z_{0,{\rm hole}} \tan k l_{\rm eff}\,, \tag{10.26}$$

with $Z_{0,{\rm hole}} = \rho c_{\rm s}/S_{\rm hole}$ the intrinsic impedance of the "tube" which is the hole.

The total air flow is the sum of the flows down the pipe and the hole:

$$U_{\rm total} = U_{\rm pipe} + U_{\rm hole} = \frac{P}{Z_{\rm pipe}} + \frac{P}{Z_{\rm hole}}\,. \tag{10.27}$$

Therefore the impedance of the instrument at the position of the tone hole is

$$Z = \frac{P}{U_{\rm total}} = \frac{P}{\frac{P}{Z_{\rm pipe}} + \frac{P}{Z_{\rm hole}}} \qquad \rightarrow \qquad \frac{1}{Z} = \frac{1}{Z_{\rm pipe}} + \frac{1}{Z_{\rm hole}}\,. \tag{10.28}$$

In other words, at the location of the tone hole,

1. Compute the impedance $Z_{\rm pipe}$ of the pipe up to the tone hole;

2. Compute the impedance $Z_{\rm hole}$ of the finger hole, considered as a tiny little pipe;

3. Add reciprocals of impedances: $1/Z$ for the tube is $1/Z$ for the pipe up to the tone hole plus $1/Z$ for the tone hole.

I emphasize again that the reason you add $1/Z$'s is because the *flow* adds: the total flow is the flow out the hole and the flow down the pipe.

To get a crude idea of what this means, approximate $kD \ll 1$ and $kl_{\rm eff} \ll 1$ (the latter is almost always true). These allow the replacements $\tan(kD) = kD$ and $\tan(kl_{\rm eff}) = kl_{\rm eff}$. Also neglect radiation losses. Then the two impedances are

$$Z_D = i\frac{\rho c_{\rm s}}{S}kD\,, \qquad Z_{\rm hole} = i\frac{\rho c_{\rm s}}{S_{\rm hole}}kl_{\rm eff}\,. \tag{10.29}$$

The total impedance is

$$Z = i(\rho c_{\rm s}k)\frac{1}{\frac{S}{D} + \frac{S_{\rm hole}}{l_{\rm hole}}}\,. \tag{10.30}$$

This can be interpreted as saying that the pipe has an "effective length"

$$Z = i(\rho c_{\rm s}k)\frac{D_{\rm eff}}{S}\,, \qquad D_{\rm eff} = \frac{1}{\frac{1}{D} + \frac{S_{\rm hole}}{Sl_{\rm eff}}}\,. \tag{10.31}$$

The pipe acts shorter than its true length, but not like it simply terminates at the location of the finger hole.

Very often the size and position of finger holes are chosen so that D/S and $l_{\rm hole}/S_{\rm hole}$ are comparable. This means that changing D actually influences the impedance of the instrument. In particular, if there are a line of finger holes, opening the hole *after* the first open hole changes the D relevant for the last hole, and so it still has some influence on the impedance of the instrument. That means that you can change the tuning of the instrument by opening or closing finger holes which are one or two past the first open finger hole.