Physics 551   Homework 11

Due Friday 21 November 2014

1   Angular pattern

Suppose a Hydrogen atom is in the $2p$ state $|n\ell m\rangle = |210\rangle$, that is, a state with zero eigenvalue of L_z. Use the explicit form of $Y_{10}(\theta, \phi)$ to show that $\langle 100| z |210\rangle$ is nonzero but $\langle 100| x |210\rangle = 0 = \langle 100| y |210\rangle$. That is, show that the transition electric dipole moment is purely in the z direction.

Based on this result, find the angular pattern of emitted photons. That is, figure out how the probability to emit a photon with wave vector $\vec{k}$ depends on the angles θ, ϕ describing $\vec{k}$'s direction in spherical coordinates. Use any shortcut you can to avoid calculating anything *other* than this information about the angular pattern. *Hint*: it all depends on the polarization vectors.

Angular momentum is conserved and the final state of the hydrogen has $L = 0$. Therefore the initial angular momentum, $|10_z\rangle$, must be carried away by the photon. Assume that a photon has spin $S = 1$, but that the component of spin along the $\hat{k}$ axis must always be $\pm\hbar$, never 0. That is, assume that a photon with wave vector $\vec{k}$ must be in an eigenstate of $\hat{k}_i S_i$ with eigenvalue $\pm\hbar$, never with eigenvalue 0. Find the sum over these two possible spin states, of the probability for the initial J-state to be found in this final J-state. Your result is a function of the direction $\hat{k}$. Show that it gives the same angular function as you found above, for the actual angular pattern of photons.

Therefore the angular pattern of emitted photons can be understood as the projection of the photon spin state onto the change in the hydrogen's L-state.

2   Hydrogen

Consider hydrogen in the $3D$ state, say, $n = 3, \ell = 2, m = 2$. What is the dominant state it will decay into, and what is the decay width? Look up the radial and angular wave functions you need in this problem. Use computer algebra with the calculations if you find it helpful.

Now consider the $4P$ state, $n = 4, \ell = 1, m = 1$. What two ℓ-states does it predominantly decay into? Why not the third possible ℓ state? What m values are possible in the final states? List all decay sequences the $4P$ state can follow, if all decays occur via $E1$ transitions – do not bother to list the possible m states, just the n, ℓ quantum numbers.

3   Microwaves and CO_2

Carbon dioxide is three atoms in a line, OCO. Oxygen is more electronegative, so the O atoms have charge $-\delta e$ and the C has charge $+2\delta e$, where δ tells what fraction of an electron the O atom "steals" from the C atom. Because the atoms are charged, some vibrational motions of

CO_2 can create an electric dipole, and therefore radiate photons. This is key to CO_2's role as a greenhouse gas, so we explore it in this problem.

There are three vibrational modes of CO_2: a "breathing" mode in which the O atoms alternately move towards and away from the C; a "sloshing" mode in which the C moves up when the O move down; and a "bending" mode in which the O move laterally in one direction and the C moves in the opposite lateral direction.

Breathing　　**Sloshing**　　**Bending**
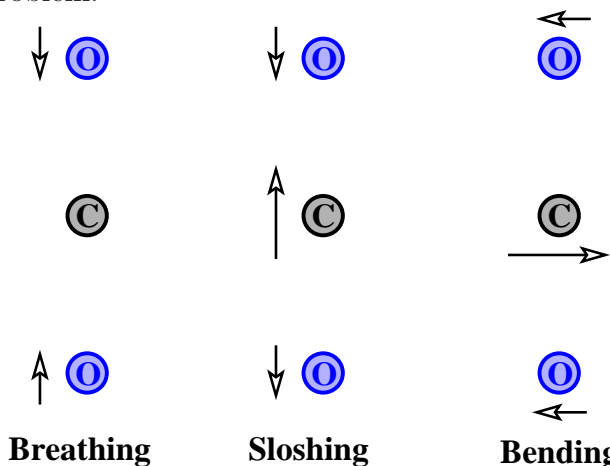
Atomic physicists quote the frequencies by writing the inverse wave length of the associated radition. They find that $\lambda^{-1} = 1388$/cm for the breathing (symmetric stretching) mode, $\lambda^{-1} = 2349$/cm for the sloshing (asymmetric stretching) mode, and $\lambda^{-1} = 666$/cm for the bending mode. [Careful: these are inverse wave lengths, not wave numbers! Think about 2π's!]

Suppose a CO_2 molecule is vibrating in the sloshing mode. Explain why there is a time-varying dipole moment (a picture may be enough). Do the same for the bending mode, and explain why there is *not* a dipole moment associated with the breathing mode.

For the sloshing mode, define x to be the distance between the C atom and the center-point of the two O atoms. Write down a Simple Harmonic Oscillator Hamiltonian for the sloshing mode, assuming that the energy varies quadratically with x:

$$H = \frac{p^2}{2\mu} + \frac{K}{2}x^2 \, . \tag{1}$$

In particular, figure out the right "reduced" mass μ, which is a function of M_O and M_C the masses of an oxygen and a carbon atom. [If you cannot figure this out, just do the rest of the problem in terms of μ. But maybe you should review your classical mechanics.] Find the spring constant K from the masses $M_O = 2.657 \times 10^{-26}$ kg and $M_C = 1.993 \times 10^{-26}$ kg, the frequency given above, and the speed of light $c = 2.9979 \times 10^8$ m/s.

Next, express the dipole moment μ_E as a function of x and δ.

Now, consider a CO_2 molecule in the first excited SHO state. What is the dipole transition matrix element $\langle 0 | \mu_E | 1 \rangle$ between the excited state and the ground state? Use this result, and the rest of the electric dipole moment formula for photon emission, to compute the decay width of the CO_2 molecule in the first vibrational state to emit a photon and fall into the ground state. Your answer will be proportional to δ^2, but all the other factors should be known. Calculate the decay lifetime in seconds, if $\delta = 0.02$.

Repeat these steps for the bending mode [very little changes].