Physics 551 Homework 12

Due Friday 5 December 2014

1 Bessel equation

In class we saw that the regular solution of the Bessel equation

$$\left[\partial_x^2 + \frac{2}{x}\partial_x + 1 - \frac{\ell(\ell+1)}{x^2}\right] R_\ell(x) = 0 \tag{1}$$

is of form

$$\lim_{x\to\infty} R_\ell(x) = \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right) . \tag{2}$$

Let us see that this is the behavior expected, for large ℓ.

First, show that the substitution $u(x) = xR(x)$ (or $R(x) = u(x)/x$) leads to a slightly simpler differential equation

$$\left[\partial_x^2 + 1 - \frac{\ell(\ell+1)}{x^2}\right] u(x) = 0 \, . \tag{3}$$

This looks like a 1-dimensional Schrödinger equation with energy $2E = 1$ and potential $2V = \ell(\ell+1)/x^2$. Next, make the approximation $\ell(\ell+1) \simeq (\ell+1/2)^2$, which is valid up to small corrections.

At what value of x is the classical turning point x_0?

Show that the WKB approximation for the solution at $x > x_0$ is

$$u(x > x_0) \;\simeq_{\mathrm{WKB}}\; \left(\frac{x^2 - x_0^2}{x^2}\right)^{-1/4} \sin\left[\frac{\pi}{4} + \int_{x_0}^{x} \sqrt{1 - x_0^2/y^2}\; dy\right] . \tag{4}$$

Here $\pi/4$ is the phase factor associated with the turning-point (Airy function) matching which we previously studied, and the integral equals (you will show!) the phase associated with the WKB approximation in the large-x region. Evaluate this expression and show that it correctly corresponds to the limiting behavior of the Bessel function

$$u_\ell(x) \rightarrow_{x\gg 1} \sin(x - \ell\pi/2) \, . \tag{5}$$

2 Small hard sphere

A model potential for short-range repulsive interactions is the hard sphere potential,

$$V(r) = \begin{cases} \infty \, , & r < r_0 \, , \\ 0 \, , & r > r_0 \, . \end{cases} \tag{6}$$

Here we consider the low-energy case in which $kr_0 \ll 1$. The scattering cross-section is dominated by the $\ell = 0$ ("s-wave") scattering channel, but for some purposes it is necessary to understand what happens in higher ℓ "waves" as well. Here we explore this.

1. Show (this is not hard) that the solution for the radial equation $R_\ell(r > r_0)$ is determined by Eq. (1), with $x \equiv kr$, subject to the boundary condition $R_\ell(x = kr_0) = 0$.

2. Show (also not hard–quote the relevant standard results from the theory of ordinary differential equations) that the most general solution to Eq. (1) is

$$R_\ell(x) = C_1 j_\ell(x) + C_2 n_\ell(x)\,, \tag{7}$$

where j_ℓ and n_ℓ are the standard solutions to the equation, the spherical Bessel and Neumann functions.

3. Show that the boundary condition fixes the ratio

$$\frac{C_2}{C_1} = -\frac{j_\ell(kr_0)}{n_\ell(kr_0)}\,. \tag{8}$$

4. By considering the asymptotic large-x limits of j_ℓ and n_ℓ,

$$j_\ell(x) \to \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)\,, \qquad n_\ell(x) \to -\frac{1}{x}\cos\left(x - \frac{\ell\pi}{2}\right)\,, \tag{9}$$

show that the phase shift is given by

$$\delta_\ell = -\arctan\frac{C_2}{C_1}\,. \tag{10}$$

5. Use the explicit expressions

$$n_0(x) = -\frac{\cos(x)}{x}\,, \qquad n_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x}\,, \tag{11}$$

and the recurrence relation

$$n_{\ell+1}(x) = \frac{2\ell+1}{x} n_\ell(x) - n_{\ell-1}(x) \tag{12}$$

to show that the leading small-x behavior of n_ℓ is

$$n_\ell(x) = -\frac{(2\ell-1)!!}{x^{\ell+1}} + \mathcal{O}(x^{-\ell})\,. \tag{13}$$

6. Define the *Wronskian* of the solutions j_ℓ, n_ℓ to be

$$W_\ell(x) \equiv j_\ell(x)\partial_x n_\ell(x) - n_\ell(x)\partial_x j_\ell(x)\,. \tag{14}$$

Consider $\partial_x W$; use the fact that j_ℓ and n_ℓ solve Eq. (1) to prove that $\partial_x W = -2W/x$ and therefore $W \propto x^{-2}$.

Show from the large-x asymptotics, Eq. (9), that $W = 1/x^2$.

7. As the regular solution to Eq. (1), $j_\ell(x) \propto x^{+\ell}$ at small x. Use the value you found for the Wronskian, together with the small-x behavior of $n_\ell(x)$, to show that

$$j_\ell(x \ll 1) = \frac{x^\ell}{(2\ell+1)!!} + \mathcal{O}(x^{\ell+1})\,. \tag{15}$$

8. Combine all your results to find the leading small-x behavior for the phase shift δ_ℓ.

Now let us apply this result. Consider scattering with $kr_0 < 1$. Show that at leading order (including only the $\ell = 0$ phase shift), the scattering is isotropic. At next order – including the $\ell = 1$ phase shift – does the forward scattering ($\theta \ll 1$) increase and the backwards scattering ($\pi - \theta \ll 1$) decrease, or vice versa?

Identical fermions must have an overall antisymmetric wave function. That means that two fermions in a symmetric spin state (for instance, spin-$\frac{1}{2}$ particles in the $S_{\text{tot}} = 1$ state) must have an antisymmetric spatial wave function. Explain why ℓ must therefore be odd. What ℓ value then dominates the scattering at small kr_0, and by what power of k – and what power of E – does the scattering shrink as k is made small?

3 Born approximation

In the Born approximation, the emission amplitude is given by

$$\begin{aligned} f(\theta,\phi) &= \frac{-m}{4\pi\hbar^2}\int V(r)e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}d^3\vec{r}\,, && (16)\\ \vec{k} &= k\hat{z}\,, && (17)\\ \vec{k}' &= k\left(\cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y}\right)\,. && (18) \end{aligned}$$

This arises at first perturbative order in V. In this problem we will explore what this has to do with the partial wave expansion. We will do this by working in from both sides.

First, let us find the phase shift at first order in V. Write down the differential equation obeyed by $R_\ell(r)$, including the contribution of V, as

$$\left(\partial_r^2 + \frac{2}{r}\partial_r + k^2 - \frac{\ell(\ell+1)}{r^2}\right)R_\ell(r) = \frac{2mV(r)}{\hbar^2}R_\ell(r)\,. \tag{19}$$

Write the regular and singular solutions of the free equation (the one where the RHS is replaced by 0) as $J(r)$ and $N(r)$ (which equal $j_\ell(kr)$ and $n_\ell(kr)$).

Argue that the solution $R_\ell(r)$ can always be written as

$$R_\ell(r) = c_j(r)J(r) + c_n(r)N(r) \tag{20}$$

and that this form is in fact underdetermined; we can add the extra condition that

$$J(r)c_j'(r) + N(r)c_n'(r) = 0\,. \tag{21}$$

(Argue this by counting freedoms and recalling that solutions to second order ODEs are determined by a value and a first derivative.) Explain why the small-r boundary condition is $c_n(0) = 0$. For simplicity choose $c_j(0) = 1$. Show that, perturbing to linear order in the potential V, that c_n obeys

$$c_n'(r) = \frac{2mV(r)}{\hbar^2}\frac{J^2(r)}{W(r)} \tag{22}$$

where $W(r) = JN' - NJ' = 1/(kr^2)$ is the Wronskian of the two solutions. By considering the definition of δ_ℓ in terms of the large-r behavior of the solution, show that

$$\delta_\ell = \lim_{r\to\infty} -\arctan\frac{c_n}{c_j}\,. \tag{23}$$

Argue that at leading order, we do not need the linear correction to c_j, so we can treat $c_j = 1$ in the above. Finally, show that to linear order in the potential,

$$\delta_\ell = \frac{-2km}{\hbar^2}\int_0^\infty V(r) r^2 J^2(r) dr\,. \tag{24}$$

Write down the expression for f_ℓ arising from this result (again to linear order in V).

Now show that this is the same as the result Eq. (16). To do so, recall that

$$f_\ell = \int_0^\infty d\phi \int_0^\pi \sin\theta\, d\theta\, Y_{\ell 0}(\theta,\phi)\, f(\theta,\phi)\,, \tag{25}$$

and use the expansion

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell'=0}^{\infty}\sqrt{4\pi(2\ell'+1)}Y_{\ell' 0}(\theta,\phi) i^{\ell'} j_{\ell'}(kr)\,, \tag{26}$$

where the angles θ,ϕ are with respect to the $\vec{k}$ axis. Make the same expansion of $e^{-i\vec{k}'\cdot\vec{r}}$ except noting that the angles are now with respect to the k' axis, which is at some angle with respect to the $\vec{k}$ axis. Perform the angular integrals using identities for spherical harmonics, and show that the resulting value of f_ℓ is the same as you find from Eq. (24). [[This may be tricky! You may need the identity relating spherical harmonics in two coordinate systems separated by an angle θ from each other:

$$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{m=\ell} Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta'',\phi'') \tag{27}$$

where θ',ϕ' and θ'',ϕ'' are the same physical point expressed in the two coordinate systems.]]