Physics 551 Homework 2

Due Friday 19 September

All problems refer to the scaled Simple Harmonic Oscillator

$$i\partial_t |\psi\rangle = h\,|\psi\rangle \;, \qquad h = \frac{1}{2}\left(\xi^2 + p_\xi^2\right) \;, \qquad p_\xi = -i\partial_\xi \tag{1}$$

and its raising and lowering operators

$$a^\dagger \equiv \frac{\xi - ip_\xi}{\sqrt{2}} \,, \quad a \equiv \frac{\xi + ip_\xi}{\sqrt{2}} \tag{2}$$

obeying the commutation relations

$$\Big[\xi\,,\, p_\xi\Big] = i\,, \qquad \Big[a\,,\, a^\dagger\Big] = 1\,. \tag{3}$$

1 SHO basics

For a state $|n\rangle$, compute $\langle\xi\rangle$, $\langle p_\xi\rangle$, $\langle\xi^2\rangle$, $\langle p_\xi^2\rangle$. Verify the Virial relation $\langle p_\xi^2\rangle = \langle\xi^2\rangle$, which shows that half the energy arises from the kinetic term and half the energy arises from the potential term. Then use your results to show that the product of the uncertainties in the energy eigenstate $|n\rangle$ is

$$\Delta\xi^2 \Delta p_\xi^2 = \frac{(2n+1)^2}{4} \,, \tag{4}$$

which exceeds the minimum value of 1/4 unless $n = 0$.

2 Coherent states

Define the operator $C(\alpha)$ and the state $|\alpha\rangle$ as:

$$C(\alpha) \equiv \exp(\alpha a^\dagger - \alpha^* a)\,, \qquad |\alpha\rangle \equiv C(\alpha)\,|0\rangle\,, \tag{5}$$

where α is a complex number.

- Show that the state is properly normalized, $\langle\alpha|\,|\alpha\rangle = 1$. (Hint: one approach is to show that $\alpha a^\dagger - \alpha^* a$ is antiHermitian, that $C(\alpha)$ is therefore unitary, and that multiplying a state by a unitary operator does not change its normalization.)
- Suppose two operators A and B obeying $\Big[A\,,\,B\Big] = c$ a complex constant. Show that $Ae^B = e^B(A + c)$. Compute $\Big[a\,,\,\alpha a^\dagger - \alpha^* a\Big]$ and $\Big[a^\dagger\,,\,\alpha a^\dagger - \alpha^* a\Big]$, and use these and the above result to show that $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$ (So $|\alpha\rangle$ is an eigenvector of a with eigenvalue α.) What is $\langle\alpha|\,a^\dagger$?

- Use the results of the last part, and the expressions for ξ and p_ξ in terms of $a, a^\dagger$, to compute $\langle\alpha|\,\xi\,|\alpha\rangle$, $\langle\alpha|\,p_\xi\,|\alpha\rangle$, $\langle\alpha|\,\xi^2\,|\alpha\rangle$, and $\langle\alpha|\,p_\xi^2\,|\alpha\rangle$. Show that $\Delta\xi^2$ and Δp_ξ^2 are the same as for the SHO ground state and saturate the uncertainty relation.

- [This may be tricky, so it will be extra credit and if you get stuck, just skip it!]

  Show that the $|\alpha\rangle$ states are not orthogonal. In particular, for $\alpha \neq \alpha'$, show that

$$\langle\alpha'|\,|\alpha\rangle\,\langle\alpha|\,|\alpha'\rangle = \exp(-|\alpha-\alpha'|^2)\,. \tag{6}$$

3 Squeezed states

Define the squeezing operator and squeezed state as

$$S(\zeta) \equiv \exp\left(\frac{\zeta a^2 - \zeta^* a^{\dagger 2}}{2}\right)\,, \qquad |\zeta\rangle \equiv S(\zeta)\,|0\rangle\,. \tag{7}$$

- Show that this state is correctly normalized, $\langle\zeta|\,|\zeta\rangle = 1$.

- Show that, for two operators A, B satisfying $\left[A\,,\,B\right] = kA$, that $Ae^B = e^B e^k A$.

- Assume that ζ is real. Find $\left[\xi\,,\,\frac{\zeta a^2 - \zeta a^{\dagger 2}}{2}\right]$ and $\left[p_\xi\,,\,\frac{\zeta a^2 - \zeta a^{\dagger 2}}{2}\right]$, and use the last result to show that

$$\xi S(\zeta) = e^{-\zeta} S(\zeta)\xi \quad \text{and} \quad p_\xi S(\zeta) = e^{\zeta} S(\zeta) p_\xi\,. \tag{8}$$

  Use this result to evaluate $\langle\xi\rangle$, $\langle\xi^2\rangle$, $\langle p_\xi\rangle$, and $\langle p_\xi^2\rangle$ in the state $|\zeta\rangle$. Show that the uncertainty relation is saturated, although the individual uncertainties are different than in the ground or coherent states.

4 Squeezed Coherent State

Consider the squeezed coherent state

$$|\alpha,\zeta\rangle = C(\alpha)S(\zeta)\,|0\rangle\,. \tag{9}$$

Assume ζ is real but allow α to be complex. Use the results and approaches of the last two problems to show that it has unit norm, that the values $\langle\xi\rangle$ and $\langle p_\xi\rangle$ are the same as for the coherent state, and that $\Delta\xi^2$ and Δp_ξ^2 are the same as for the squeezed state.

Which of these answers would change if you considered the state $S(\zeta)C(\alpha)\,|0\rangle$?

5 Poisson distribution

The binomial distribution

$$P_{\text{Binomial}}(n, N, x) = x^n (1-x)^{N-n} \frac{N!}{n!(N-n)!} \tag{10}$$

is the probability of something happening n times, if there are N opportunities for it to happen and each has probability x. The Poisson probability distribution

$$P_{\text{Poisson}}(n, y) = \frac{y^n}{n!} e^{-y} \tag{11}$$

is the limit of P_{Binomial} in which $N \to \infty$ while $x \to 0$, keeping the average number of events $y = Nx$ fixed. That is, it tells how many times an event will happen, if there are a huge number of individually unlikely chances.

Verify the following properties of the Poisson distribution:

1. The (discrete) convolution of two Poisson distributions is a Poisson distribution,

$$\sum_{m=0}^{n} P_{\text{Poisson}}(m, x) P_{\text{Poisson}}(n-m, y) = P_{\text{Poisson}}(n, x+y)\,. \tag{12}$$

2. It is correctly normalized,

$$\langle 1 \rangle = \sum_{n=0}^{\infty} P_{\text{Poisson}}(n, y) = 1 \tag{13}$$

3. The average value of n is y:

$$\langle n \rangle = \sum_{n=0}^{\infty} n P_{\text{Poisson}}(n, y) = y \tag{14}$$

4. The mean squared value of n is

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 P_{\text{Poisson}}(n, y) = y^2 + y \tag{15}$$

Use these results, and the result (shown in class) that $|\langle n|\,|\alpha\rangle|^2 = P_{\text{Poisson}}(n, \alpha^* \alpha)$, to find the expectation value of energy in a coherent state, $\langle \alpha | \, h \, | \alpha \rangle$, and the mean squared fluctuations in energy $\sigma_h^2 = \langle \alpha | \, h^2 \, | \alpha \rangle - (\langle \alpha | \, h \, | \alpha \rangle)^2$.