Physics 551 Homework 3

Due Friday 26 September

The early problems again refer to the scaled Simple Harmonic Oscillator

$$i\partial_t \left|\psi\right\rangle = h\left|\psi\right\rangle \,, \qquad h = \frac{1}{2}\left(\xi^2 + p_\xi^2\right) \,, \qquad p_\xi = -i\partial_\xi \tag{1}$$

and its raising and lowering operators

$$a^\dagger \equiv \frac{\xi - ip_\xi}{\sqrt{2}} \,, \quad a \equiv \frac{\xi + ip_\xi}{\sqrt{2}} \tag{2}$$

obeying the commutation relations

$$\left[\xi \,,\, p_\xi\right] = i \,, \qquad \left[a \,,\, a^\dagger\right] = 1 \,. \tag{3}$$

1 Energy fluctuations

Consider the coherent state $\left|\alpha\right\rangle = C(\alpha)\left|0\right\rangle$ and the squeezed state $\left|\zeta\right\rangle = S(\zeta)\left|0\right\rangle$, with the same definitions for the operators as in the previous homework set. Consider general complex values for the parameters α and ζ.

Compute directly $\left\langle\alpha\right| h \left|\alpha\right\rangle$, $\left\langle\alpha\right| h^2 \left|\alpha\right\rangle$, $\left\langle\zeta\right| h \left|\zeta\right\rangle$, and $\left\langle\zeta\right| h^2 \left|\zeta\right\rangle$. Use these to determine the uncertainty in the energy, for each case (coherent and squeezed), as a function of α or ζ.

Hint: although α, ζ are complex, the expectation values are time independent. Is there a more convenient choice of time to evaluate them? It may be easiest to work with $h = (\xi^2 + p_\xi^2)/2$ rather than expressing it in terms of $a, a^\dagger$; and use how ξ, p_ξ "move past" the operators $C(\alpha), S(\zeta)$ as derived in the last homework set.

extra credit: compute $\langle h\rangle$ and $\langle h^2\rangle$ for a squeezed coherent state, with α, ζ generic (so in particular there is no time when they are simultaneously real).

2 How to get a coherent state

Consider the *time dependent* Hamiltonian which arises when a simple harmonic oscillator is subject to a time-dependent external force $F(t)$:

$$H(t) = \frac{\xi^2 + p_\xi^2}{2} - F(t)\xi \,. \tag{4}$$

Suppose that there is no force before some time t_0, so $F(t < t_0) = 0$; and assume that the system begins in the ground state $\left|0\right\rangle$ at time $t = t_0$. Show that, at time $t_1 > t_0$, the system will be in the coherent state $e^{i\phi(t)}\left|\alpha(t_1)\right\rangle$, with

$$\alpha(t_1) = \frac{i}{\sqrt{2}} \int_{t_0}^{t_1} e^{i(t-t_1)} F(t)\, dt \,, \tag{5}$$

and $\phi(t)$ some time-dependent phase which you should also compute. *Hint*: show that the proposed explicit expression solves the Schrödinger equation with the time dependent force.

3 Parity and eigenfunctions

Consider quantum mechanics in one dimension under a *confining potential*, that is, $V_1(x)$ which diverges to $+\infty$ as $\to \infty$ or $x \to -\infty$. The spectrum of eigenvalues of the Hamiltonian

$$H_1 = -\frac{\hbar^2}{2m}\partial_x^2 + V_1(x) = \frac{p^2}{2m} + V_1(x) \tag{6}$$

is discrete and the states are properly normalizable.[1]

When a continuous real function changes sign, it must pass through zero. The point where it equals zero is called a *node*. In this problem we will show that the n'th energy eigenstate has $n-1$ nodes, and that, if $V(x) = V(-x)$ so that there is a parity symmetry, that the eigenfunctions are alternately even and odd under parity.

If you get stuck on any subproblem, just ASSUME the result is true and tackle the remaining subproblems.

1. Write the equation satisfied by an energy eigenfunction $u_{\mathrm{E}}(x)$ with energy E. Show that, if $u_{\mathrm{E}}(x)$ is a solution, then $u_{\mathrm{E}}^*(x)$ is also a solution, with the same energy. Use this to show that all eigenfunctions may be taken to be purely real. (This should be easy.)

2. Show that, if $H_1 = p^2/2m + V_1(x)$ and $H_2 = p^2/2m + V_2(x)$ with $V_2(x) \geq V_1(x)$ for all x, then $\langle\psi| H_1 |\psi\rangle \leq \langle\psi| H_2 |\psi\rangle$ for any normalizable finite-energy state $|\psi\rangle$ (it need not be an energy eigenstate of either Hamiltonian). That is, a state's energy is always lower under H_1 than under H_2.

3. Use this result to show that the ground state energy under H_1 is lower than the ground state energy under H_2.

4. Suppose the $u_{\mathrm{E}}(x)$ is an energy eigenfunction of H_1 with a node at $x = x_0$. Then defining

$$V_2(x) = \begin{cases} V_1(x) & x < x_0\,, \\ \infty & x > x_0\,, \end{cases} \tag{7}$$

   and

$$u(x) = \begin{cases} u_{\mathrm{E}}(x) & x < x_0 \\ 0 & x > x_0 \end{cases}\,, \tag{8}$$

   show that $u(x)$ is an energy eigenfunction of H_2 with the same energy as $u_{\mathrm{E}}(x)$ has under H_1.

   Repeat but taking $u(x)$ and $V_2(x)$ to be finite for $x > x_0$ and zero/infinity for $x < x_0$.

5. Now suppose that $u_{\mathrm{E}}(x)$ has two nodes at x_0, x_1. Show that the function $u(x) = u_{\mathrm{E}}(x)$ on the interval $[x_0, x_1]$ and $u(x) = 0$ elsewhere is an energy eigenstate of H_2, with the same energy as $u_{\mathrm{E}}(x)$ has under H_1, if we now define $V_2(x) = V_1(x)$ in the interval $[x_0, x_1]$ and $V_2(x) = \infty$ elsewhere.

   Show that in each of these cases, $u(x)$ is *not* an energy eigenfunction of the original H_1.

[1] We could prove this, but assume in this problem that it has already been proven.

6. Use 4 to prove that the ground state under H does not have any nodes. *Hint*: use contradiction. If the ground state had a node, show that there is a state with an equal or lower energy which is not an energy eigenstate ...

7. Prove that the ground state is the *only* energy eigenstate with no nodes. (*Hint*: orthogonality)

8. Suppose that $u_2(x)$ and $u_3(x)$ are energy eigenfunctions with energies $E_2 < E_3$. Show that

   - the first node of $u_2(x)$ is at larger x than the first node of $u_3(x)$.
   - the last node (the node with largest x) of $u_2(x)$ occurs at a smaller value of x than the last node of $u_3(x)$;
   - If x_3, x_4 are neighboring nodes of $u_3(x)$, there are never two nodes x_1, x_2 of $u_2(x)$ between them, eg, with $x_3 < x_1 < x_2 < x_4$.

   Hint: for the first two, use 4. For the last, use 5. In each case, find two potentials V_2 and V_3 which each equal V_1 in some range and are infinite outside, and for which the (restricted) functions $u_{2,3}$ are the ground states.

9. Suppose $u_m(x)$, $u_n(x)$ are distinct eigenfunctions of H_1 with m, n nodes. Show that $E_m < E_n$ implies that $m < n$.

10. Show that the spectrum of H_1 is nondegenerate; that is, there are never linearly independent energy eigenfunctions u_1, u_2 with the same energy. Hint: where would their nodes be?

11. Prove that, if $V(x) = V(-x)$, the Hamiltonian commutes with the parity operator π.

    Use this to show that the energy eigenfunctions must be eigenfunctions of parity, with eigenvalue ± 1, that is, $\pi \left| u_{\mathrm{E}} \right\rangle = \pm \left| u_{\mathrm{E}} \right\rangle$. Show that functions of eigenvalue $+1$ always have an even number of nodes and functions of eigenvalue -1 always have an odd number of nodes. Therefore the eigenfunctions are alternately of even and odd parity, with an even-parity ground state.