Physics 551 Homework 4

Due Friday 3 October

1 Runge-Lenz Vector

Consider the Hamiltonian of a particle in a central potential of $1/r$ form:

$$H = \frac{p_i p_i}{2m} - \frac{Ze^2}{r} \tag{1}$$

where $r = \sqrt{r_i r_i}$. First consider the problem classically: write down Hamilton's equations of motion and show that they conserve both the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ (or if you prefer, $L_{ij} = r_i p_j - r_j p_i$) and the Runge-Lenz vector

$$\vec{M} \equiv \frac{\vec{p} \times \vec{L}}{m} - \frac{Ze^2}{r}\vec{r}\,. \tag{2}$$

Now consider the quantum theory. Define $M_i = \epsilon_{ijk}(p_j L_k + L_k p_j)/2m - (Ze^2/r)r_i$ (note operator ordering), and verify **ANY THREE OF THE FOUR** following relations:

$$\begin{aligned}
\vec{L} \cdot \vec{M} &= 0\,, \\
\Big[L_i\,,\, H\Big] &= 0 = \Big[M_i\,,\, H\Big]\,, \\
\Big[M_i\,,\, L_j\Big] &= i\hbar\epsilon_{ijk} M_k\,, \\
M^2 &= \frac{2H}{m}\Big(L^2 + \hbar^2\Big) + Z^2 e^4\,,
\end{aligned}$$

(We will skip verifying commutation relations of two M's, which is messy.) *Hint*: work in index notation, not vector notation, and remember that $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

Finally, when we introduce $N_i = M_i\sqrt{-m/2E}$ the commutation relations become [don't show this]

$$\begin{aligned}
\Big[L_i\,,\, L_j\Big] &= i\hbar\epsilon_{ijk} L_k\,, \\
\Big[N_i\,,\, L_j\Big] &= i\hbar\epsilon_{ijk} N_k\,, \\
\Big[N_i\,,\, N_j\Big] &= i\hbar\epsilon_{ijk} L_k\,.
\end{aligned}$$

Show that, introducing

$$I_i \equiv \frac{L_i + N_i}{2}\,, \qquad K_i \equiv \frac{L_i - N_i}{2}\,, \tag{3}$$

that $\vec{I}$ and $\vec{K}$ obey the commutation relations [this should be easy]

$$\begin{aligned}
\Big[I_i\,,\, I_j\Big] &= i\hbar\epsilon_{ijk} I_k\,, \\
\Big[K_i\,,\, K_j\Big] &= i\hbar\epsilon_{ijk} K_k\,, \\
\Big[I_i\,,\, K_j\Big] &= 0\,.
\end{aligned}$$

2 Hydrogen Hamiltonians

Consider the Following Hamiltonian for a hydrogenic atom. The electron and nucleus have spin, with spin operators $\vec{S}_e$ and $\vec{S}_n$ respectively. As usual $p_i \equiv -i\hbar\partial_i$.

$$\begin{aligned}
H &= H_{\rm kin} + H_{\rm coul} + H_{\rm rel} + H_{\rm S-O} + H_{\rm Weak} + H_{\rm EDM} \,, \\
H_{\rm kin} &= \frac{p^2}{2m} \,, \\
H_{\rm coul} &= -\frac{e^2}{r} \,, \\
H_{\rm rel} &= -\frac{p^2p^2}{8m^3c^2} - \frac{1}{8m^2c^2}\left[p_i \,, \left[p_i, -e^2/r\right]\right] \,, \\
H_{\rm S-O} &= \frac{e^2}{2m^2c^2r^3}\epsilon_{ijk}r_ip_jS_{e,k} \,, \\
H_{\rm Weak} &= Q_{\rm weak}\frac{e^{-M_Z cr/\hbar}}{mr}\vec{S}_e\cdot\vec{p} \,, \\
H_{\rm EDM} &= -\frac{e\mu_E}{r^3}\vec{r}\cdot\vec{S}_n \,.
\end{aligned}$$

Here the terms represent the electron kinetic term, the Coulomb interaction, relativistic corrections, the relativistic spin-orbit interaction, a possible weak interaction with strength given by the dimensionless quantity Q and range limited by the Z-boson mass M_z, and a nuclear electric dipole moment interaction.

Check that each term is Hermitian. Explain why the last term can be interpreted as the nucleus possessing an electric dipole moment. Determine the transformation properties of each term under parity and under time reversal symmetry. Which term or terms would you try to detect experimentally to study the possibility that Nature violates parity? Which term or terms would you try to detect to study the possibility that Nature violates time-reflection symmetry?

3 Tensor versus Vector Notation

In class we considered rotation matrices

$$R_{ij} = \exp(\theta_{ij}) \tag{4}$$

and their corresponding operators

$$\mathcal{D}(R) = \exp\left(\frac{i}{2\hbar}J_{ij}\theta_{ij}\right) \,. \tag{5}$$

Argue that if we define

$$\theta_i = -\frac{1}{2}\epsilon_{ijk}\theta_{jk}$$

then θ_z corresponds to the angle of a rotation about the z-axis as it is usually defined. Show that defining

$$J_i = \frac{1}{2}\epsilon_{ijk}J_{jk} \,, \tag{6}$$

that

$$\mathcal{D}(R) = \exp\left(-\frac{i}{\hbar}\theta_i J_i\right) \tag{7}$$

which is the notation for the rotation operator which you are accustomed to. Finally, show that when we define J_i as above, that the commutation relations we found for the J_{ij},

$$\Big[J_{ij}\,,\,J_{lm}\Big] = i\hbar\Big(\delta_{mj}J_{il} - \delta_{lj}J_{im} + \delta_{im}J_{lj} - \delta_{il}J_{mj}\Big) \tag{8}$$

correctly provide the commutation relations for the J_i,

$$\Big[J_i\,,\,J_j\Big] = i\hbar\epsilon_{ijk}J_k\,. \tag{9}$$

4 Commutation of position and angular momentum

We saw that the commutation relations between momentum and angular momentum are

$$\Big[p_i\,,\,J_j\Big] = i\epsilon_{ijk}p_k\,. \tag{10}$$

This relation should ensure that

$$\mathcal{D}^\dagger(R)p_i\mathcal{D}(R) = R_{ij}p_j\,, \tag{11}$$

with R_{ij} the rotation matrix. Show that this is true for *infinitesimal* rotations about the z-axis, by checking how p_x, p_y, and p_z behave when $\mathcal{D}(R)$ is an infinitesimal z-rotation matrix.

Next, we might guess from the fact that p_i and x_i are both vectors that

$$\Big[x_i\,,\,J_j\Big] = i\epsilon_{ijk}x_k\,. \tag{12}$$

To verify this, first prove that the Jacobi identity,

$$\Big[A\,,\,[B,C]\Big] + \Big[B\,,\,[C,A]\Big] + \Big[C\,,\,[A,B]\Big] \tag{13}$$

is true for any three operators A, B, C (*Hint*: expand all commutators explicitly and show that all terms cancel). Then apply the Jacobi identity to the three operators x_i, p_j, and J_k:

$$\Big[x_i\,,\,[p_j,J_k]\Big] + \Big[J_k\,,\,[x_i,p_j]\Big] + \Big[p_j\,,\,[J_k,x_i]\Big] = 0 \tag{14}$$

and show that the identity holds if Eq. (12) is true, but it does not hold if $\Big[x_i\,,\,J_j\Big]$ is proportional to p or to J.