Physics 551 Homework 5

Due Friday 10 October

1 Reducible representations

Consider two sources of angular momentum: orbital L and spin S. Take a particle to be spin-$\frac{1}{2}$ and in the $\ell = 1$ orbital state. Assume $\left[L_i\,,\,S_j\right] = 0$ and define as usual $J_i = L_i + S_i$.

- Write down L_x, L_y, L_z, S_x, S_y, and S_z as matrices in the basis $|\ell m_\ell, s_z\rangle = |11,+\rangle$, $|11,-\rangle$, $|10,+\rangle$, $|10,-\rangle$, $|1-1,+\rangle$, $|1-1,-\rangle$.
- Write J_x, J_y, J_z in this basis.
- Write J^2 in this basis. Is it diagonal?
- Find the similarity transform S such that $S^\dagger J_i S$ are block diagonal, with J^2 diagonal and equal to $3\hbar^2/4$ in the upper and $15\hbar^2/4$ in the lower entries. Write $J_{x,y,z}$ and J^2 in this basis.

  Based on your result, how many of what irreducible representations make up the $1 \otimes \frac{1}{2}$ (reducible) representation?

Now do the problem the easier way. *Hint:* if you do this first, it will be much easier to find the similarity transformation matrix S. Do *not* use matrix notation in the following section, but work in terms of kets which are linear combinations of $|1m_\ell, \pm\rangle$.

- Verify explicitly that $|11,+\rangle$ is an eigenstate of J^2 and of J_z with eigenvalue $15\hbar^2/4$ and $3\hbar/2$ respectively. Name this state $|3/2, 3/2\rangle$.
- Find the state $J_- |11,+\rangle$ explicitly. Find the properly normalized version, and name it $|3/2, 1/2\rangle$.
- Find the properly normalized state which is a linear combination of the same basis elements as $|3/2, 1/2\rangle$ but is orthogonal to it. Verify that this state is an eigenstate of J^2 and J_z with eigenvalues $3\hbar^2/4$ and $\hbar/2$ respectively.
- Continue to act with J_- on each of these states to find the rest of each definite-J^2 multiplet.

The coefficients on the states in each multiplet can be used to read off the similarity transformation matrix S.

2 Three spins

For a system of three spin-$\frac{1}{2}$ particles, with spin operators S_1, S_2, S_3, find the allowed values of the total spin $S^2 = (S_1 + S_2 + S_3)^2$ and the number of different states with each spin. (This problem should be very easy.)

3 Hamiltonians with spin

In mass resonance imaging (MRI), an intense static magnetic field in the z direction and a smaller, time-varying field in the x, y directions is applied to a sample containing spins. The spin Hamiltonian in these fields is

$$i\hbar\partial_t |\psi\rangle = -\vec{\mu} \cdot \vec{B} |\psi\rangle = -\mu\vec{S} \cdot \vec{B} |\psi\rangle = (AS_z + B(S_x \cos\omega t + S_y \sin\omega t)) |\psi\rangle \ . \tag{1}$$

Here $\vec{\mu}$ is the magnetic moment, which is proportional to spin with proportionality μ; A and B are coefficients which incorporate the information about μ and the strengths of the different components of the magnetic field.

If a spin is $|+\rangle$ at $t = 0$, find the probability that it will be found in the $|+\rangle$ state at a later time t. In the case that $A \gg B$, what condition should ω satisfy, so that the probability of a transition to the $|-\rangle$ state is large? *Hint*: you can write the state as $f(t) |+\rangle + g(t) |-\rangle$. But it may be useful to define $\tilde{g}(t) = e^{-i\omega t} g(t)$, that is, to work in the basis $|+\rangle$ and $e^{-i\omega t} |-\rangle$.

(By making B_z and therefore A vary through a sample, this allows one to flip spins in a small region of the sample; therefore one can determine the density of spins in that region, allowing imaging.)

4 Spin analogues in particle physics

Strong interactions are invariant (up to small corrections which we will ignore in this problem) under rotations in an internal "isotopic spin" space, whose generators T_1, T_2, T_3 satisfy the same algebra as the angular momentum generators J_x, J_y, J_z. Therefore the theory of representations of isospin is identical to that for spin.

The proton and neutron are the $|+\rangle$ and $|-\rangle$ states of an isospin-$\frac{1}{2}$ doublet: $T_+ |n\rangle = |p\rangle$. The three pions $|\pi^+\rangle$, $|\pi^0\rangle$, and $|\pi^-\rangle$ form an isospin triplet, with $T_3 |\pi^+\rangle = +|\pi^+\rangle$. The nuclei $|{}^3\mathrm{He}\rangle$ and $|{}^3\mathrm{H}\rangle$ form a doublet, and the deuteron $|D\rangle$ is a singlet.[1] Here "singlet" means (iso)spin-0, "doublet" means (iso)spin-$\frac{1}{2}$, and "triplet" means (iso)spin-1 (the terminology reflects the number of states in the (iso)spin multiplet).

Consider the reactions

$$p + D \to \pi^0 + {}^3\mathrm{He}\,, \qquad p + D \to \pi^+ + {}^3\mathrm{H}\,. \tag{2}$$

The initial state combines a particle of (iso)spin $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and a particle of (iso)spin $|0, 0\rangle$, so its total (iso)spin and T_z component of isospin are $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$. If the interactions respect isospin symmetry, the final state must also have total isospin and T_z isospin projection of $\frac{1}{2}$ and $\frac{1}{2}$. This lets us determine the ratio of the rates for these processes; the final state must be

$$\begin{aligned} |f\rangle &= \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = a\left|1, 1; \tfrac{1}{2}, \tfrac{-1}{2}\right\rangle + b\left|1, 0; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle \\ &= a\left|\pi^+, {}^3\mathrm{H}\right\rangle + b\left|\pi^0, {}^3\mathrm{He}\right\rangle \ . \end{aligned}$$

[1] A triplet state, featuring a two-proton state, an excited state of the deuteron, and a two-neutron state, is not bound.

Use angular momentum summation methods to determine a and b, and find the ratio of probabilities for the two final states by examining the ratio of the squares of these amplitudes.

Next, suppose that when a π scatters with an n or p, that only the isospin-3/2 part of the wave function participates.[2] In this case, predict the ratios of the following processes:

$$\begin{aligned} &\pi^+ + p \to \pi^+ + p\,, \quad \pi^- + p \to \pi^- + p\,, \quad \pi^- + p \to \pi^0 + n\,, \\ &\pi^+ + n \to \pi^+ + n\,, \quad \pi^+ + n \to \pi^0 + p\,, \end{aligned} \tag{3}$$

Hint: in each case the amplitude for the process involves the product of the projection of the initial state on the appropriate isospin-3/2 state times the projection of the final state on the isospin-3/2 state. The relevant (iso)spin combinations are exactly the ones you examined to death in the first problem.

5 Limit of large angular momentum

In this problem we will check a few ways that the theory of angular momentum reduces to the expected behavior in the limit of large J^2. The problem is not too hard, don't make it harder than it is!

1. Classically, if we add two angular momentum vectors of length J_1 and J_2, what range of values can the sum $J^2 = (\vec{J}_1 + \vec{J}_2)^2$ take? Assuming that the relative angle is uniformly random over the sphere, what is the probability distribution of final values of J^2? Does this correspond with the behavior in quantum mechanics, for large $j_1/\hbar$ and $j_2/\hbar$?

2. Consider $\hat{n}(\theta)$ to be the unit vector in the x, z plane, an angle θ from the z axis (so $\hat{n}(0) = \hat{z}$ and $\hat{n}(\pi/2) = \hat{x}$).

   Find $\left[J_z\,,\,J_{\hat{n}(\theta)}\right]$. Consider the state $|j,j\rangle$ with maximum J_z eigenvalue. Find the mean and mean squared values of $J_{\hat{n}(\theta)}$. Is the relation between $\langle J_{\hat{n}(\theta)}\rangle$, j, and θ the one you would expect classically? For large j, is the uncertainty $\Delta J_{\hat{n}(\theta)}$ small compared to $\langle J_z\rangle$ for all θ?

[2]In general this is not the case, but at an energy equal to the mass of an isospin-3/2 particle [the Δ resonance], this will be the case.