Physics 551 Homework 6

Due Friday 17 October

1 Building Spherical Tensors

Consider two vector operators U_i, V_i satisfying

$$\begin{aligned}\Big[J_i\,,\,U_j\Big] &= i\hbar\epsilon_{ijk}U_k\\ \Big[J_i\,,\,V_j\Big] &= i\hbar\epsilon_{ijk}V_k\\ \Big[U_i\,,\,V_j\Big] &= 0\,.\end{aligned}$$

(That is, U_i, V_i are each a triplet of operators which satisfy commutation relations like those of the momentum operator p_i).

Define

$$U_1 \equiv \frac{-U_x - iU_y}{\sqrt{2}}\,, \qquad U_0 \equiv U_z\,, \qquad U_{-1} \equiv \frac{+U_x - iU_y}{\sqrt{2}}\,, \tag{1}$$

and define $V_{1,0,-1}$ similarly. Show that these obey

$$\begin{aligned}\Big[J_z\,,\,U_m\Big] &= m\hbar U_m\,,\\ \Big[J_\pm\,,\,U_m\Big] &= \hbar\sqrt{2 - m(m\pm 1)}\,U_{m\pm 1}\,,\end{aligned}$$

and show that this is the same as how J_z and $J_\pm$ act on a spin-1 state, $J_{z,\pm}\,|1,m\rangle$. Therefore these operators act, under commutation with angular momentum operators, the same way a spin-1 state acts under multiplication by angular momentum operators. For this reason we will name this a "spin-1 spherical tensor" (notation $U^{(1)}_m$ where the (1) superscript says it acts like it is spin-1). [It may be most convenient to do the remainder of the problem using the basis $U_{1,0,-1}$ rather than $U_{x,y,z}$.]

Name the operator $U_iV_j \equiv T_{ij}$ (a rank-2 tensor). Consider the operator U_1V_1. Show that it obeys

$$\Big[J_z\,,\,U_1V_1\Big] = 2\hbar U_1V_1\,, \qquad \Big[J_+\,,\,U_1V_1\Big] = 0\,, \tag{2}$$

the same as the state $|2,2\rangle$ when multiplied by J_z, J_+. Therefore we will name it $T^{(2)}_2$.

Find $T^{(2)}_1$, defined as

$$\Big[J_-\,,\,T^{(2)}_2\Big] = NT^{(2)}_1 \tag{3}$$

with the normalization N chosen such that $\Big[J_+\,,\,T^{(2)}_1\Big] = NT^{(2)}_2$. [What is N?] This is the same process we used to get the state $|2,1\rangle$ from the state $|2,2\rangle$. Find the linear combination of the operators U_0V_1 and U_1V_0 which is orthogonal to $T^{(2)}_1$ and name it $T^{(1)}_1$. Show that $\Big[J_+\,,\,T^{(1)}_1\Big] = 0$. Show that $\Big[J_z\,,\,T^{(2)}_1\Big] = \hbar T^{(2)}_1$ and $\Big[J_z\,,\,T^{(1)}_1\Big] = \hbar T^{(1)}_1$, so the lower index indeed describes the J_z value.

Continue this procedure, in complete analogy with the procedure for combining two $j=1$ states into $j=2,1,0$ states, to find $T^{(2)}_{0,-1,-2}$, $T^{(1)}_{0,-1}$, and $T^{(0)}_0$.

Finally, write out all 9 of these operators in terms of $U_{x,y,z}V_{x,y,z}$ (for instance, you should find that $T_0^{(0)} = (U_xV_x + U_yV_y + U_zV_z)/\sqrt{3}$.) Show that the $T_m^{(2)}$ are symmetric on $U \leftrightarrow V$ and traceless, while $T_m^{(1)}$ are antisymmetric, while $T_0^{(0)} = U_iV_i/\sqrt{3}$ is the trace of T_{ij}.

2 Invariant tensors

For any rotation matrix $R \in \mathrm{SO}(3)$ (the proper rotation group, orthogonal matrices of unit determinant), show that

$$\sum_{i'j'} R_{ii'}R_{jj'}\delta_{i'j'} = \delta_{ij} \qquad \text{and} \qquad \sum_{i'j'k'} R_{ii'}R_{jj'}R_{kk'}\epsilon_{i'j'k'} = \epsilon_{ijk}\,. \tag{4}$$

Use this to show that a rank-2 tensor T_{ij} and a rank-3 tensor X_{ijk}, contracted with δ_{ij} and with ϵ_{ijk}

$$\sum_{ij} T_{ij}\delta_{ij} \qquad \text{and} \qquad \sum_{ijk} X_{ijk}\epsilon_{ijk} \tag{5}$$

each transform as a scalar. (Here you should consider δ_{ij} and ϵ_{ijk} as invariant symbols rather than as tensors, that is, do not perform a rotation on their indices, only on the indices of T_{ij} and X_{ijk}.)

Consider the product of 2 spin-1 representations and of 3 spin-1 representations. In each case determine how many of what irreducible representations arise. In particular, how many spin-0 representations arise? You should find one spin-0 representation for the case of 2 spin-1's and 3 spin-1's – these correspond to the contractions discussed above.

3 Angular momentum, Vectors, 3D SHO

Consider a spinless particle in a Simple Harmonic Oscillator potential in three dimensions:

$$H = \frac{p_ip_i}{2m} + \frac{k}{2}r_ir_i\,, \tag{6}$$

where $i = x, y, z$ (or $1, 2, 3$) and Einstein summation convention is used. Define $\omega = \sqrt{k/m}$ as usual. Note (don't bother to show!) that $\Big[H\,,\,J_i\Big] = 0$, so angular momentum is conserved in this problem.

Define a triple of operators a_x, a_y, a_z and their daggers,

$$a_x = \sqrt{\frac{m\omega}{2\hbar}}\left(r_x + \frac{ip_x}{m\omega}\right)\,, \qquad a_y\,,\; a_z \quad \text{defined similarly.} \tag{7}$$

They obey $\Big[a_i\,,\,a_j\Big] = 0$ and $\Big[a_i\,,\,a_j^\dagger\Big] = \delta_{ij}$ (don't bother proving this).

Show that the triple a_i form a vector, in the sense that $\Big[a_i\,,\,J_j\Big] = i\hbar\epsilon_{ijk}a_k$. Show that $a_i^\dagger$ is also a vector. Also show that a_i is parity-odd.

What are the possible energies for the eigenstates of H?

Prove that the minimum energy state is unique, parity-even, and spin-0. Show that the eigenstates obtained by acting on the vacuum (minimum energy) state with an odd number of $a^\dagger$ raising operators are always parity-odd, while states obtained by acting with an even number of $a^\dagger$ operators are parity even. Given the parity behavior of angular momentum, what can we say, already, about the possible ℓ values of these states?

The product of n creation operators is a totally symmetric rank-n Cartesian tensor. What is the maximum angular momentum ℓ one can achieve by acting with n creation operators, and what other values of ℓ are possible? (It is harder to show on symmetry principles alone, but each of these states appears in the spectrum and none are duplicated, so for instance, there are not two independent sets of spin-2 states created by 4 creation operators.)

If we wanted to find all energy eigenstates with angular momentum ℓ more directly, we could insert $R(r)Y_{lm}(\theta,\phi)$ into the time-independent Schrödinger equation and find a radial equation for $rR(r)$. Find this radial equation (feel free to look this up in an introductory text). You should find a 1-dimensional time-independent Schrödinger equation with a centrifugal barrier added to the potential. Show for $\ell = 0$ that the properties of the 1-dimensional simple harmonic oscillator correctly explain the energies of spin-0 states you found above. Next consider large but finite ℓ. Find the r where the potential is minimized, the value of the potential at the minimum, and the second derivative of the potential at the minimum. Approximate the potential as a simple harmonic oscillator about this minimum, and determine the minimum energy and state spacing, all to lowest order in $1/\ell$. Show that the minimum energy corresponds to the energy where angular momentum ℓ states actually do first enter the spectrum, and that the spacing is the same as the spacing of angular momentum ℓ states which you found above.