Physics 551 Homework 7

Due Friday 24 October 2014

1 Gaussian integral technicalities

In class we used the integral

$$\int_{-\infty}^{\infty} \exp(-iAp^2/2)\, dp = \sqrt{\frac{2\pi}{iA}}\,. \tag{1}$$

Here we see to what extent this is really true.

First, show that the integral as written is not absolutely convergent. Nevertheless, we can define $\int_{-R}^{R} \exp(-iAp^2/2)\, dp$ and hope that the $R \to \infty$ limit exists.

Next, show that at large R, the contribution from a line in the complex p plane from R to $(1-i)R$ gives a negligible integral. Use Cauchy's theorem to show that the integral can then be performed along the line in the complex p plane, from $p = (i-1)R$ to $p = (1-i)R$. Evaluate this integral and show that the $R \to \infty$ limit is completely well behaved.

2 Practice with density matrices

Consider two spin-$\frac{1}{2}$ particles with spin operators S_1, S_2; the total spin is $S = S_1 + S_2$. We can either write states in terms of the individual spins, with states of form $|{+}{+}\rangle$ or $|{+}{-}\rangle$, or in terms of the total spin, with states of form $|sm\rangle$, eg, $|11\rangle$ or $|00\rangle$.

Work throughout in the basis $\{|{+}{+}\rangle\ |{+}{-}\rangle\ |{-}{+}\rangle\ |{-}{-}\rangle\}$. That is, express a general state $|\psi\rangle$ as a column vector

$$\psi_a = \begin{bmatrix} \psi_{++} \\ \psi_{+-} \\ \psi_{-+} \\ \psi_{--} \end{bmatrix}, \qquad \psi_{++} = \langle{+}{+}|\,|\psi\rangle\ , \quad \text{etc.} \tag{2}$$

First, find the column vector expression for the state $|11\rangle$. Then find the column vector expressions for $|10\rangle$ and for $|00\rangle$ (the states of vanishing S_z and total spin 1 and 0).

Next, write the density matrix corresponding to each of these (pure) states. Check (this is trivial) that they each have trace 1.

Now suppose that the second spin cannot be observed (it gets lost or escapes). Trace over its value to find the reduced density matrix for the first spin, in each of the three cases. Show that each density matrix remains trace-1; but the $|11\rangle$ density matrix gives rise to a pure state while the other two are mixed states (probability distributions). With the second spin lost, is there *any* way to tell whether the initial state was $|10\rangle$ or $|00\rangle$?

It might strike you as peculiar that $|11\rangle$ turns into a pure state but $|10\rangle$ does not, even though they start out as states in the same multiplet. This is because the state $|10\rangle$ is really structurally different than the state $|11\rangle$ – not because it has a different expectation value of

S_z. To see this, consider the state $|11_x\rangle$ obeying $S_x|11_x\rangle = \hbar\,|11_x\rangle$. (You get to it from $|11\rangle$ by applying a $\pi/2$ rotation about the y axis.) Find an explicit expression for the state $|11_x\rangle$ in our chosen basis [for instance, by applying a rotation operator to the state $|11\rangle$, or by already knowing the explicit form of $|+_x\rangle$ – use the easiest way you can find, don't be fancy]. Show that

$$\langle 11_x|\, S^2\, |11_x\rangle = 2\hbar^2\,, \qquad \langle 11_x|\, S_z\, |11_x\rangle = 0 \tag{3}$$

(just like the state $|10\rangle$). Now, find the density matrix associated with this state. Integrate out the second spin (assume it cannot be observed or measured) and find the density matrix for the remaining spin. Show that it corresponds to a pure state [though it is not diagonal in the basis we have chosen].