Physics 551 Homework 8 Solutions

1 Density matrices and orthogonality

A spin-$\frac{1}{2}$ particle is in a statistical ensemble with a 50% probability to be in the $|+_z\rangle$ state (the eigenstate of S_z with eigenvalue $\hbar/2$) a 50% chance to be in $|+_x\rangle$ (the eigenstate of S_x with eigenvalue $\hbar/2$). [Note that these states are not orthogonal. Don't worry about that yet.] Use the standard procedure,

$$\rho = \sum_{|\psi_i\rangle} P_i \, |\psi_i\rangle \, \langle\psi_i| \tag{1}$$

to write this density operator in terms of states in the $|\pm_z\rangle$ basis, and then as a matrix using this basis. Use the density matrix to compute the probability that a measurement of the z-component of spin will return value $+\hbar/2$.

Now solve the eigenvalue/eigenvector problem for the density matrix. The eigenstates you find are eigenstates of spin along a definite axis – that is, eigenstates of $\vec{S} \cdot \hat{n}$ for some unit vector $\hat{n}$. Find $\hat{n}$. What is the entropy of the density matrix, and is it the same as you would guess at the beginning, knowing that there is a 50% chance to be in each of two states? What would be a more proper description of the density matrix, in terms of probabilities to be in orthogonal states?

2 Thermal density matrix

The thermal density operator is

$$\rho = \frac{\exp(-H/T)}{\mathrm{Tr}\ \exp(-H/T)} \,. \tag{2}$$

(The denominator just ensures that the trace of ρ is 1.) Write an explicit expression for the density operator of a simple harmonic oscillator with energy spacing $\hbar\omega$. Find $\langle n \rangle$ (that is, $\langle a^\dagger a \rangle$) and the entropy.

Next consider the Hamiltonian of a two-state quantum system with states $|+\rangle$ and $|-\rangle$ and Hamiltonian $H = E_0 \, |+\rangle \, \langle +|$. Write an explicit expression for the thermal density operator and find the probability to be in the $|+\rangle$ state and the entropy.

3 Landau Levels

The early parts of this problem are Problem 2.39 from the book.

Consider an electron moving in a uniform magnetic field in the z-direction, $\vec{B} = B\hat{z}$. Defining as usual

$$\Pi_x \equiv p_x - \frac{eA_x}{c} \,, \qquad \Pi_y \equiv p_y - \frac{eA_y}{c} \,, \tag{3}$$

determine the commutator $\left[\Pi_x \,, \, \Pi_y\right]$.

Write down the Hamiltonian H, and the difference between the Hamiltonain and the part associated only with z-motion, $H_{xy} \equiv H - p_z^2/2m_e$. Argue from the form of H_{xy} and the commutation relations you found (and by drawing an analogy with the previously solved problem of a single simple harmonic oscillator), that the energy eigenstates of H_{xy} are discrete. Show that the allowed energies are

$$E_{xy} = \frac{|eB|\hbar}{m_e c}\left(n + \frac{1}{2}\right) , \qquad E = E_{xy} + \frac{k_z^2}{2m_e} , \tag{4}$$

where k is the eigenvalue of the p_z operator. The states with different values of n are called *Landau levels*. [You may recognize $\omega = |eB|/m_e c$ as the cyclotron frequency of an electron in this magnetic field.]

In the case of the SHO, we know that the ground state is nondegenerate. It then follows that each excited state is also nondegenerate. But for the problem we are studying here, it is not obvious – and in fact, not true – that the ground state of H_{xy} is nondegenerate. Let us find its degeneracy.

First, show that, if there are precisely N independent eigenstates of H_{xy} with energy eigenvalue $(n + 1/2)(|eB|\hbar/m_e c)$, then there are also precisely N independent eigenstates with energy $(n + 3/2)(|eB|\hbar/m_e c)$.

Next, suppose that the magnetic field B is uniform in a box of length L_x by L_y. Use the usual physics trick of treating this box as periodic. For the case $B = 0$, use the standard physics tricks to find the approximate total number of states with $E_{xy} < E_0$, where E_0 is some specific, very large value. Give a plausible physical argument that the number of states up to energy E_0 should not depend much on the value of B. Considering next the case where B is not zero, find the approximate number of H_{xy} eigenstates up to energy E_0 in terms of B and N the degeneracy. Equate these relations to find the degeneracy N as a function of L_x, L_y, and B. Show that N only depends on the total magnetic *flux* $\Phi_B = BL_xL_y$. Compare the degeneracy N to the number of Dirac flux quanta.