Physics 551 Homework 9

Due Friday 7 November 2014

1 Landau Levels and Spin

(This problem should be pretty easy. If not, you are probably doing something wrong!) Revisit the last problem in the previous homework assignment by including the fact that electrons have a half-integer spin, which is also associated with a magnetic moment.

Show that a classical, uniform-density, uniformly charged, spinning ball with a charge-to-mass ratio of e/m has a dipole-moment-to-angular-momentum ratio of

$$\frac{\mu}{J} = \frac{e}{2mc}\,. \tag{1}$$

Hint: the answer is the same for a uniform-density, uniform-charge spinning ring [since a ball is built out of a whole bunch of rings].

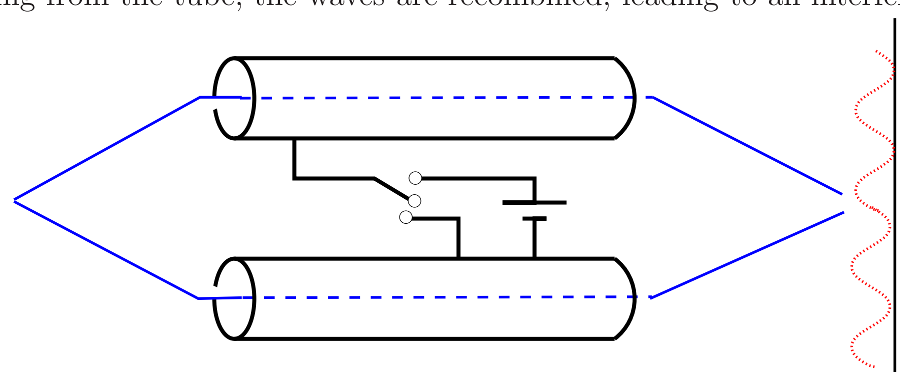
If an electron followed this rule, we would have $\vec{\mu} = \frac{e}{2mc}\vec{S}$. The ratio for a fundamental particle with spin need not follow this rule, so it is customary to define a fudge-factor g: $\vec{\mu} = (ge/2mc)\vec{S}$. Now add the spin-magnetic energy to the Hamiltonian of the previous problem,

$$H = H_{xy} - \frac{geB}{2mc}S_z + \frac{p_z^2}{2m}\,. \tag{2}$$

What are the allowed energies and degeneracies of $H - p_z^2/2m = H_{xy} - \frac{geB}{2mc}S_z$, for general g? What about for the actual value for electrons, which is $g = 2$?

2 Space-time version of Aharonov-Bohm

An electron's wave function is split between two paths. Each path enters a hollow conducting tube. Emerging from the tube, the waves are recombined, leading to an interference pattern.



Suppose that, while the electrons are in the tube, an electric field is established between the tubes such that there is a potential difference of $\Delta\phi$, for a time Δt. The electric field is turned on after the electron enters the tube and turned off before it emerges; and the tubes act as Faraday cages so there is never any electric field inside. Therefore the electron never experiences any electric field.

What value must the product $\Delta\phi\Delta t$ take, such that the final interference pattern shifts by one half of one fringe? Calculate this two ways: once by considering how the phase develops under the Schrödinger equation, and once by thinking about the contribution of the gauge potential to the path integral. Show that the phase difference is the integral of the electric flux on the (space-time) surface bounded by the two electron paths. Therefore this interference effect is analogous to the Aharonov-Bohm effect with magnetic fields.

3 Time-independent Perturbation Theory

This problem is to make sure you remember how to do time-independent perturbation theory. It also finishes some details from Homework 6 problem 3. Indeed, where possible copy your answers from your Homework 6 results.

In Homework 6 we encountered the Hamiltonian (after rescaling to eliminate all dimensionful quantities)

$$H = \frac{(-i\partial_x)^2}{2} + V(x)\,, \qquad V(x) = \frac{x^2}{2} + \frac{\ell(\ell+1)}{2x^2}\,. \tag{3}$$

Consider the case where ℓ is large but not infinite.

1. Find the location of the minimum of the potential x_0, and the value at the minimum $V(x_0)$. Expand in large ℓ, keeping the $\mathcal{O}(1/\ell)$ term but dropping ℓ^{-2} and higher terms.

2. Find $V''(x_0)$. Use the Taylor expansion to second order about x_0 to treat the problem as a Simple Harmonic Oscillator, and show that the ground state energy is

$$E_0 = \ell + \frac{3}{2} - \frac{1}{8\ell} + \mathcal{O}(\ell^{-2})\,. \tag{4}$$

   This *differs* from the correct answer, $\ell + 3/2$.

3. Find the next three terms in the Taylor expansion of the potential (to order $(x-x_0)^5$). Show that the odd-power terms do not contribute at first order. Show that the linear energy pertubation from the $(x-x_0)^4$ term and the quadratic energy perturbation from $(x-x_0)^3$ are both of order $1/\ell$, while all higher-power terms can be neglected.

4. Calculate these $1/\ell$ perturbations to the ground state energy and show that they cancel the $-1/8\ell$ term found above, yielding the correct ground state energy.

5. Compute the $1/\ell$ perturbations (linear order from $(x-x_0)^4$ and quadratic order from $(x-x_0)^3$) to the n'th energy level and show that they still precisely cancel the $-1/8\ell$ term from the minimum of the potential to yield the correct level spacing.

4 Actually solving the above problem

Now let us actually solve the problem with Hamiltonian

$$H = \frac{1}{2}\left(-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^2\right) . \tag{5}$$

First, find the ground state energy. Hint: just guess that the ground state wave function is

$$\psi(x) = x^m e^{-x^2/2} , \tag{6}$$

and stuff this form into the time-independent Schrödinger equation,

$$H\psi = E\psi . \tag{7}$$

Is there a value of m (probably ℓ-dependent) for which this is a solution? What is the E value? Prove that this wave function is the ground state [by quoting a result from a previous homework]. Name this state $|0\rangle$ and its energy E_0.

Next, show that the energy levels are evenly spaced. Do this by defining three operators:

$$\begin{aligned} H_- &= \frac{1}{2}\left(-\partial_x^2 + \frac{\ell(\ell+1)}{x^2}\right) = \frac{1}{2}\left(p^2 + \frac{\ell(\ell+1)}{x^2}\right) = H - x^2/2 \\ H_+ &= \frac{x^2}{2} , \\ D &= \frac{xp+px}{2} = -i\frac{x\partial_x + \partial_x x}{2} \end{aligned}$$

where in the definition of D, the last ∂_x acts on x and anything to its right, as usual. Determine the three commutators:

$$\Big[H_- \, , \, H_+\Big] , \qquad \Big[H_- \, , \, D\Big] , \qquad \Big[H_+ \, , \, D\Big] . \tag{8}$$

Now define

$$L_\pm \equiv H_- - H_+ \pm iD . \tag{9}$$

Show that

$$\Big[H \, , \, L_\pm\Big] = \pm 2L_\pm , \qquad \Big[L_- \, , \, L_+\Big] = 4H . \tag{10}$$

Show that $L_- |0\rangle = 0$ (explicitly). Show that the state $|n\rangle \equiv L_+^n |0\rangle$ has nonzero norm and energy $E_0 + 2n$.

[To really finish the problem we should find the correct normalization for these states and we should prove that each operation by L_+ adds one node to the wave function, so we have all states. But let's not bother.]