QUANTUM MECHANICAL PURE STATES WITH GAUSSIAN WAVE FUNCTIONS

Bonny L. SCHUMAKER

Theoretical Astrophysics 130–33, California Institute of Technology, Pasadena, California 91125, U.S.A.

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Abstract:

This paper examines single-mode and two-mode Gaussian pure states (GPS), quantum mechanical pure states with Gaussian wave functions. These states are produced when harmonic oscillators in their ground states are exposed to potentials, or interaction Hamiltonians, that are linear or quadratic in the position and momentum variables (i.e., annihilation and creation operators) of the oscillators. The physical and group theoretical properties of these Hamiltonians and the unitary operators they generate are discussed. These properties lead to a natural classification scheme for GPS. Important properties of single-mode and two-mode GPS are discussed. An efficient vector notation is introduced, and used to derive many of the important properties of GPS and of the Hamiltonians and unitary operators associated with them.

1. Introduction and overview

This paper considers Gaussian pure states (GPS), quantum mechanical pure states that have Gaussian wave functions (i.e., wave functions that are exponentials of complex-valued linear and quadratic forms in "position" or "momentum" variables). These states are particularly relevant to the description of a harmonic oscillator with a nearly classical, coherent excitation whose intrinsic quantum mechanical fluctuations are important. Such a description arises, for example, in connection with the transmission or detection of coherent optical signals [1, 2], or high-precision measurements of a macroscopic oscillator's displacement, as in the detection of gravitational waves [3, 4]. Gaussian pure states are familiar in quantum optics, where they describe the coherent output from a laser and the predicted "squeezed-state" light [5, 6] from an optical parametric amplifier. For the theorist, these states have the satisfying feature that the Hamiltonians for the physical processes that produce them are known and have simple, easily interpreted forms.

Gaussian pure states are produced when harmonic oscillators in their ground states are subjected to particular kinds of time-dependent potentials, or interaction Hamiltonians. The oscillators might be mechanical or electrical, or they might be the normal (bosonic) modes of a quantized field such as the electromagnetic field; for purpose of illustration, I shall have the last of these examples in mind throughout this paper. The interaction Hamiltonians that produce GPS are polynomials that are linear and/or quadratic in the oscillators' position and momentum variables. Hence, although they may affect N oscillators ($N \geq 1$), thus producing an N-mode GPS, the interaction Hamiltonians are sums of Hamiltonians that either involve a single oscillator or couple two oscillators. This has the important consequence that one need look only at single-mode and two-mode GPS in order to understand the fundamental features of all N-mode GPS. Single-mode GPS and their subsets have been studied by many people during the last six decades [7–16]. The goal of this paper is to help make the less widely known and understood two-mode GPS as familiar as their single-mode counterparts.

This first section, composed of nine subsections (1.1 through 1.9), provides an introduction to and overview of the results derived in the remainder of the paper. Subsection 1.1 establishes the logical connection between the Gaussian nature of the wave functions and the Hamiltonians for the physical processes that produce GPS. The nature of these Hamiltonians, and of the unitary ("evolution") operators they generate, is described briefly. Subsection 1.2 describes in detail the interaction Hamiltonians, and classifies them according to their important physical properties. Subsection 1.3 goes on to describe the unitary operators associated with the different interaction Hamiltonians. Subsection 1.4 begins a discussion of Gaussian pure states. The notion of the "total noise" of a state is defined, and a general classification scheme for GPS, based on their total noise, is introduced. Subsection 1.5 reviews briefly the definition of coherent states [10, 11]. Subsection 1.6 introduces the reader to "single-mode squeezed states" [12, 13, 17] (or "two-photon coherent states" [16]), the most general kind of single-mode GPS. Subsection 1.7 defines the most general kind of two-mode GPS. Subsection 1.8 discusses squeezing in the context of two-mode GPS, with emphasis on a special set of two-mode GPS – the "two-mode squeezed states" [18–21]. The natural physical variables for characterizing squeezing in two-mode states – the "quadrature-phase amplitudes" [18–21] – are motivated and defined briefly. Subsection 1.9 provides a brief outline of the remainder of the paper.

1.1. Introduction

Associated with any oscillator is a real, positive, constant frequency Ω. The quantum mechanical free Hamiltonian for the oscillator is

$$H_0^{(1)} \equiv \Omega a^\dagger a\,, \qquad \Omega = \Omega^*\,, \tag{1.1a}$$

where a and $a^\dagger$ are annihilation and creation operators for the mode ($[a, a^\dagger] = 1$). (Here and throughout this paper I use units with $\hbar \equiv c \equiv 1$.) The expectation value of $a^\dagger a$, the photon-number operator for the mode, is the number of photons in the mode. The free Hamiltonian for N oscillators is the sum of N single-mode free Hamiltonians:

$$H_0^{(N)} \equiv \sum_{j=1}^{N} \Omega_j a_j^\dagger a_j\,, \qquad \Omega_j = \Omega_j^*\,. \tag{1.1b}$$

The stationary states for each oscillator (eigenstates of $H_0^{(1)}$) are the number states $|n\rangle$,

$$|n\rangle \equiv (n!)^{-1/2} (a)^\dagger)^n |0\rangle\,. \tag{1.2a}$$

$$H_0^{(1)}|n\rangle = n\Omega|n\rangle\,, \tag{1.2b}$$

where the state vector $|0\rangle$ represents the ground state. Throughout this paper the state vector $|0\rangle$, or the term "vacuum state", means the tensor product of the ground states of N oscillators, for any $N \geq 1$. The vacuum state, unlike the other number states, is an eigenstate of the annihilation operators for all the modes. Its wave function is Gaussian, whereas the wave functions for the other number states $|n\rangle$, $n \geq 1$, are not [22].

The forms of the interaction Hamiltonians that produce (or preserve) Gaussian pure states are derived in this paper by considering the most general single-mode and two-mode Gaussian wave functions, in which all parameters are arbitrary, subject to normalization. The wave functions imply that Gaussian pure states are eigenstates of certain kinds of linear combinations of creation and annihilation operators. These linear combinations in turn determine the general form of the unitary operators that relate Gaussian pure states to the vacuum state, in the following way. Let the most general (normalized) N-mode GPS be expressed formally as the state vector $U_g^{(N)}|0\rangle$, where $U_g^{(N)} \equiv \exp[-iH_g^{(N)}t]$ is a unitary operator with Hermitian (self-adjoint) generator $H_g^{(N)}$. (Physically, $H_g^{(N)}$ represents a Hamiltonian and therefore has dimensions of energy; t has dimensions of (energy)$^{-1}$, or time, in units with $\hbar \equiv c \equiv 1$.) Since the vacuum state is an eigenstate (with zero eigenvalue) of the annihilation operators a_j, $j = 1, 2, \ldots, N$, an

N-mode GPS is an eigenstate (with zero eigenvalue) of the transformed annihilation operators $g_j \equiv U_g^{(N)} a_j U_g^{(N)\dagger}$. The Gaussian nature of the wave functions implies that the operators g_j are linear combinations of annihilation and creation operators plus constants, which in turn implies that the Hermitian generator $H_g^{(N)}$ consists only of linear and bilinear combinations of annihilation and creation operators. There are no further restrictions on the generator $H_g^{(N)}$, so $H_g^{(N)}$ consists, in general, of all possible Hermitian linear and bilinear combinations of annihilation and creation operators.

The unitary operators $\hat{U}_g^{(N)} \equiv \exp[-i(H_g^{(N)} t)]$ that relate N-mode GPS to the vacuum state factor naturally into unitary operators whose generators are (Hermitian) linear combinations of creation and annihilation operators, and unitary operators whose generators are (Hermitian) bilinear combinations of creation and annihilation operators. There are N unitary operators whose generators are linear in creation and annihilation operators, one for each mode, and they are identical to each other in form. They are called "displacement operators" [10]. In contrast, the unitary operators whose generators are bilinear combinations of creation and annihilation operators divide into four basic types, which differ fundamentally from each other in both their physical and group theoretical significance. In this paper they are referred to as rotation, mixing, single-mode squeeze, and two-mode squeeze operators. This division reflects the underlying structure of the $N(2N+1)$-dimensional Lie algebra associated with all bilinear combinations of N creation and annihilation operators. These unitary operators and their generators are described below.

The proof (for $N = 1$ and $N = 2$) that the unitary operator $U_g^{(N)}$, whose generator $H_g^{(N)}$ is a sum of all linear and bilinear combinations of creation and annihilation operators, factors into a product of displacement, rotation, mixing, and squeeze operators is subsumed by more general proofs given in subsections 2.3, 3.3, and appendix A. There each term in the generators $H_g^{(1)}$ and $H_g^{(2)}$ is allowed to have an arbitrary time dependence (subject to overall Hermiticity), and the unitary evolution operator $U^{(N)}(t)$, the solution to the Schrödinger equation $i\partial_t U^{(N)}(t) = H_g^{(N)}(t) U^{(N)}(t)$, $\quad U^{(N)}(0) = 1$, $(N = 1, 2)$, is shown to factor into a product of these unitary operators. The Hermitian forms associated with the displacement, rotation, mixing and squeeze operators thus take on a physical meaning, in addition to their group theoretical roles. When allowed to take on time dependences, they become the Hamiltonians for all the physical processes that produce (or preserve) Gaussian pure states. Their properties are now described.

1.2. Interaction Hamiltonians

The interaction Hamiltonians that produce Gaussian pure states divide naturally into two broad categories: those that conserve the total number of photons in the mode(s), and those that do not. Those that conserve total photon number leave the vacuum state unchanged, and their effect on other GPS is merely to redistribute the photons among the different modes. Of greater interest here are those interactions that do not conserve the total photon number, but that do preserve the Gaussian nature of a state. As stated above, all interaction Hamiltonians that produce or preserve GPS are polynomials that are linear and/or quadratic in creation and annihilation operators (i.e., in position and momentum variables). Conversely, all such interaction Hamiltonians describe physical processes that produce or preserve Gaussian states. Those that conserve the total photon number must consist of products of equal numbers of creation and annihilation operators. The requirement that they also preserve Gaussians implies that they have the (normally-ordered) forms

$$H_R^{(N)}(t) \equiv \sum_{i,j=1}^{N} \Pi_{ij}(t) a_i^\dagger a_j , \qquad \Pi_{ij} = \Pi_{ji}^* , \tag{1.3}$$

where the $\Pi_{ij}(t)$ are arbitrary complex-valued functions of time t. In contrast, the Hamiltonians that produce or preserve GPS but do not conserve the total number of photons have the forms

$$H_1^{(N)}(t) \equiv \sum_{j=1}^{N} \mathrm{i}\lambda_j^*(t)a_j - \mathrm{i}\lambda_j(t)a_j^\dagger , \tag{1.4a}$$

$$H_2^{(N)}(t) \equiv \sum_{i,j=1}^{N} \tfrac{1}{2}\mathrm{i}\zeta_{ij}^*(t)a_i a_j - \tfrac{1}{2}\mathrm{i}\zeta_{ij}(t)a_j^\dagger a_i^\dagger , \tag{1.4b}$$

where $\lambda_j(t)$ and $\zeta_{ij}(t)$ $[=\zeta_{ji}(t)]$ are arbitrary complex-valued functions of time.

The photon number-conserving interactions defined in eq. (1.3) divide naturally into two types. The first is made up of the terms for which $i=j$; for each mode i these Hamiltonians have the form

$$H_{\mathrm{R}}^{(1)}(t) = \omega_i(t)a_i^\dagger a_i , \qquad \omega_i(t) = \omega_i^*(t) , \tag{1.5}$$

which looks like the free Hamiltonian for the mode but with a time-dependent real function (not restricted to positive values) in place of the frequency. These Hamiltonians conserve the number of photons in each mode; hence they conserve the total energy, as well as the total number of photons. They are referred to in this paper as rotation Hamiltonians. Like the free Hamiltonian, they cause a time-dependent exchange of kinetic and potential energy within each mode, but unlike the free Hamiltonian, the time dependence need not be harmonic.

The second type of photon number-conserving interaction is made up of the terms in eq. (1.3) for which $i \neq j$. For each pair of modes i and j, these Hamiltonians have the form

$$H_{\mathrm{R}ij}(t) \equiv \Pi_{ij}(t)a_i^\dagger a_j + \Pi_{ij}^*(t)a_j^\dagger a_i , \qquad i \neq j . \tag{1.6}$$

These interactions conserve the total number of photons in each pair of modes, but not the number in each mode separately; i.e., the Hamiltonians $H_{\mathrm{R}ij}(t)$ commute with the sum, but not the difference, of the photon-number operators for the two modes. Physically, they describe "ideal" frequency-converting interactions, in which a photon of frequency $\Omega_j \leq \Omega_i$ and a "pump" photon of (or photons of total) frequency $\Omega_i - \Omega_j$ are destroyed simultaneously to produce a photon of frequency Ω_i, and vice-versa. The interaction is "ideal" if the pump(s) can be assumed to have an unlimited supply of photons, and so be described by a classical function $\Pi_{ij}(t)$.

The interactions defined in eq. (1.4), which do not conserve photon number, are of three types. The first consists of the linear Hamiltonians $H_1^{(1)}(t)$, each of which describes the interaction of an oscillator with a classical force characterized by a function $\lambda(t)$ (e.g., a classical current) [10, 11]. These will be seen to conserve quantities that describe the quantum noise (uncertainties) associated with GPS. The second type of interaction consists of those quadratic Hamiltonians $H_2^{(2)}(t)$ that couple two different modes ($i \neq j$). For a pair of modes i, j these Hamiltonians have the form

$$H_{2ij}(t) \equiv \mathrm{i}\zeta^*(t)a_i a_j - \mathrm{i}\zeta(t)a_j^\dagger a_i^\dagger , \qquad i \neq j . \tag{1.7}$$

These interactions conserve the difference in the number of photons in the two modes, but not the total number; i.e., the Hamiltonians $H_{2ij}(t)$ commute with the difference, but not the sum, of the photon-number operators for the two modes [23]. Physically, they describe "ideal" nondegenerate two-photon interactions, in which two photons of frequencies Ω_i and Ω_j are destroyed simultaneously to produce a pump photon of (or photons of total) frequency $\Omega_i + \Omega_j$, and vice-versa. The simplest example of a device that operates by such an interaction is a nondegenerate parametric amplifier [23–25], which uses a single pump at frequency $\Omega_i + \Omega_j$; the two modes are called the signal and the idler. Another example is a four-wave mixer [26, 27], which uses two pumps, the sum of whose frequencies is $\Omega_i + \Omega_j$. The interaction is "ideal" if the pump(s) can be described by a classical function $\zeta(t)$.

The third type of interaction that does not conserve photon number consists of the quadratic Hamiltonians $H_2^{(1)}(t)$ that involve single modes ($i = j$). For each mode these Hamiltonians have the form

$$H_2^{(1)}(t) = \tfrac{1}{2}\mathrm{i}\zeta^*(t)a^2 - \tfrac{1}{2}\mathrm{i}\zeta(t)a^{\dagger 2}\,. \tag{1.8}$$

Physically, these interactions describe ideal degenerate two-photon interactions in which two photons of frequency Ω from the same mode are destroyed simultaneously to produce a pump photon of (or photons of total) frequency 2Ω, and vice-versa. Such an interaction is used, for example, in a degenerate parametric amplifier. For specificity throughout the remainder of this paper, whenever I need an example of a device that operates by a two-photon interaction (degenerate or nondegenerate), I shall have in mind the simplest example – an ideal parametric amplifier.

1.3. Unitary operators

The unitary operators that relate one GPS to other GPS with the same total number of photons are generated by the photon number-conserving Hermitian forms $H_{\mathrm{R}}{}^{(N)}$. They are of two types: rotation operators, which act on one mode at a time, and "mixing" operators, which couple two modes. For each mode a rotation operator $R(\theta)$ is defined by

$$R(\theta) \equiv \exp(-\mathrm{i}\theta a^\dagger a)\,, \qquad \theta = \theta^* \tag{1.9}$$

[eq. (1.5)]. Formally, $R(\theta)$ rotates the real and imaginary parts of a (i.e., position and momentum) into each other. For each pair of modes i, j a mixing operator $T(q, \chi)$ is defined by

$$T(q, \chi) \equiv \exp[q(\mathrm{e}^{-2\mathrm{i}\chi} a_j^\dagger a_i - \mathrm{e}^{2\mathrm{i}\chi} a_i^\dagger a_j)]\,, \qquad i \neq j \tag{1.10a}$$

[eq. (1.6)], where q and χ are real numbers defined on the intervals

$$0 \le q \le \tfrac{1}{2}\pi\,, \qquad -\tfrac{1}{2}\pi < \chi \le \tfrac{1}{2}\pi\,. \tag{1.10b}$$

Formally, $T(q, \chi)$ unitarily transforms a_i and a_j into linear combinations of each other.

The unitary operators that relate one GPS to other GPS with different total photon number are generated by the (non-photon-number-conserving) Hermitian forms $H_1{}^{(N)}$ and $H_2{}^{(N)}$. Again, they are of two types: those that act on one mode at a time, and those that couple two modes. For each mode a displacement operator [10, 11] and a single-mode squeeze operator [12, 13, 17] are defined by

$$D(a, \mu) \equiv \exp[\mu a^\dagger - \mu^* a]\,, \tag{1.11}$$

$$S_1(r, \varphi) \equiv \exp[\tfrac{1}{2}r(\mathrm{e}^{-2\mathrm{i}\varphi}a^2 - \mathrm{e}^{2\mathrm{i}\varphi}a^{\dagger 2})] \tag{1.12a}$$

[eqs. (1.4a), (1.8)]. Here μ is a complex number, and r and φ, known as the squeeze factor and squeeze angle, are real numbers defined on the intervals

$$0 \leq r < \infty, \qquad -\tfrac{1}{2}\pi < \varphi \leq \tfrac{1}{2}\pi. \tag{1.12b}$$

Formally, the displacement operator adds a constant (μ) to a, thus changing the mean values of the position and momentum variables. The single-mode squeeze operator mixes a with $a^\dagger$. Consequently, it induces a correlation between the position and momentum variables that is independent of their mean values. This correlation can result, for example, in a narrowing of the coordinate-space wave function, with corresponding broadening of the momentum-space wave function.

For each pair of modes i, j a two-mode squeeze operator [18–21] $S(r, \varphi)$ is defined by

$$S(r, \varphi) \equiv \exp[r(\mathrm{e}^{-2\mathrm{i}\varphi} a_i a_j - \mathrm{e}^{2\mathrm{i}\varphi} a_j^\dagger a_i^\dagger)], \qquad i \neq j \tag{1.13}$$

[eq. (1.7)], where r and φ are defined as above [eq. (1.12b)]. The two-mode squeeze operator mixes a_i with $a_j^\dagger$, and a_j with $a_i^\dagger$. Consequently, it induces correlations between the positions and momenta of the two modes (but not of each mode, as the single-mode squeeze operator would do); i.e., it causes a_i and a_j to become correlated.

1.4. Gaussian pure states (GPS)

Turn now from discussion of the interaction Hamiltonians and unitary operators associated with GPS to the states themselves. Although it is useful to classify the interaction Hamiltonians and unitary operators according to whether they conserve the total number of photons, it is not so useful to classify the states according to their total number of photons. More useful for classifying GPS is a quantity that ignores the mean excitation of each mode ($\langle a_j \rangle$, $j = 1, 2, \ldots, N$) and focuses exclusively on the total (second-moment) noise associated with the state. The total noise of a single-mode GPS is defined as the mean-square uncertainty in a, the sum (hence the adjective "total") of the squared uncertainties (variances) in the real and imaginary parts of a. The minimum total noise allowed by quantum mechanics (i.e., by the commutator $[a, a^\dagger] = 1$) for each mode is one half quantum ("zero-point noise"). This minimum is realized if and only if the state is an eigenstate of the annihilation operator for that mode. The total noise of an N-mode GPS is defined as the sum of the contributions from ("total noises" of) each mode. The total noise of a GPS can be thought of as the noise content of the state in units of photon number; it is the number of photons, including a half quantum from each mode due to zero-point noise, that would be left in the state if the mean excitations were removed. The total noise of a state is a more fundamental quantity than the total number of photons. It is conserved if the total number of photons is conserved, but the converse is not true. [For example, a classical force interacting with an oscillator(s) changes the total number of photons, but not the total noise; see subsection 2.1.3.]

When considering two or more modes one should note the distinction between the total noise and another quantity, the total noise energy. The total noise energy of a GPS is the noise content of the state in units of energy; it is the energy, including zero-point energy, that would be left in the state if the mean excitations were removed. For a single mode the distinction is not important, since the total noise energy is equal to the product of the total noise and the mode's frequency. But for two or more modes with different frequencies, the total noise and the total noise energy need not be proportional to each other. (They are proportional to each other only when the total noises of all the modes are identical.) Just as photon number is a more convenient quantity than energy for classifying the potentials that produce GPS, so total noise is a more convenient quantity than total noise energy for classifying GPS.

It is shown below that all linear interaction Hamiltonians $H_1^{(N)}(t)$, as well as all photon number-conserving interaction Hamiltonians $H_R^{(N)}(t)$, conserve the total noise of an N-mode state. Further, these are the only interaction Hamiltonians that conserve both the total noise and the Gaussian nature of a state. This means that states unitarily related to each other by products of rotation, mixing, or displacement operators all have the same total noise. Conversely, all GPS with the same total noise are related to each other by (products of) rotation, mixing, and displacement operators. Only the quadratic, non-photon-number-conserving potentials $H_2^{(N)}(t)$ can change the total noise of a state. There are, therefore, two broad classes of GPS. The first class consists of all states unitarily related to the vacuum state by products of displacement, rotation, and mixing operators. These states have a total noise equal to that of the vacuum state, the minimum allowed by quantum mechanics ($\frac{1}{2}N$, for an N-mode state). Put another way, the first class consists of all (normalized) eigenstates of annihilation operators. The second class consists of all states unitarily related to states in the first class by products of single-mode and/or two-mode squeeze operators. The total noise of these states is necessarily greater than that of the vacuum state.

1.5. Coherent states

The single-mode GPS produced when an oscillator in its ground state is acted on by a classical force, i.e., subjected to the linear interaction Hamiltonian $H_1^{(1)}(t)$, is called a single-mode coherent state [10, 11]. Formally, a single-mode coherent state, symbolized by the state vector $|\mu\rangle_{\text{coh}}$, is defined as that state unitarily related to the vacuum state by the single-mode displacement operator,

$$|\mu\rangle_{\text{coh}} \equiv D(a, \mu)|0\rangle\,. \tag{1.14}$$

It is an eigenstate of the annihilation operator a with eigenvalue μ. An N-mode coherent state is simply a tensor product of N single-mode coherent states. For example, a two-mode coherent state, symbolized by the state vector $|\boldsymbol{\mu}\rangle_{\text{coh}}$ (or $|\mu_+, \mu_-\rangle_{\text{coh}}$), is defined by

$$\begin{aligned}|\boldsymbol{\mu}\rangle_{\text{coh}} &\equiv |\mu_+, \mu_-\rangle_{\text{coh}} \equiv |\mu_+\rangle_{\text{coh}}|\mu_-\rangle_{\text{coh}} \\ &\equiv D(a_+, \mu_+)D(a_-, \mu_-)|0\rangle \equiv D(\boldsymbol{a}, \boldsymbol{\mu})|0\rangle\end{aligned} \tag{1.15}$$

(the two modes are labeled here and henceforth by "+" and "−"). It is an eigenstate of the annihilation operators a_+ and a_- for the two modes, with eigenvalues μ_+ and μ_-, respectively. All normalized N-mode states that are eigenstates of the annihilation operators for their modes can be described as N-mode coherent states. That is, all states unitarily related to a coherent state by products of rotation, displacement, and mixing operators can be described as another coherent state, with different eigenvalues. Glauber [10, 11] and others [23, 28–32] beginning in the early 1960s have used coherent states to build a powerful description of the electromagnetic field. Today these states are at the heart of quantum optics, providing the basis for a sophisticated theory of the laser, for example.

1.6. Single-mode squeezed states

For a single mode, there is only one interaction Hamiltonian, $H_2^{(1)}(t)$, that can produce a GPS whose total noise differs from (i.e., is greater than) that of a coherent state. The state produced when an oscillator in a coherent state is subjected to an interaction described by $H_2^{(1)}(t)$ is called a "single-mode squeezed state" [12, 13, 16, 17] (SMSS). Formally, a SMSS, symbolized by the state vector $|\mu_\alpha\rangle_{(r,\varphi)}$, is defined as that state unitarily related to the single-mode coherent state $|\mu_\alpha\rangle_{\mathrm{coh}}$ by the single-mode squeeze operator,

$$|\mu_\alpha\rangle_{(r,\varphi)} \equiv S_1(r,\varphi)|\mu_\alpha\rangle_{\mathrm{coh}} . \tag{1.16}$$

The SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ is an eigenstate of the "single-mode squeezed annihilation operator" [18–20]

$$\alpha(r,\varphi) \equiv S_1(r,\varphi) a S_1^{\dagger}(r,\varphi) , \tag{1.17}$$

with complex eigenvalue μ_α. Any state unitarily related to the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ by products of single-mode rotation, displacement, and squeeze operators can be expressed as another SMSS $|\mu'_\alpha\rangle_{(r',\varphi')}$ (multiplied by an unobservable overall phase factor), with different squeeze factor, squeeze angle, and eigenvalue. Any unitary operator U whose generator is a linear combination $H_g^{(1)}$ of the Hermitian forms $H_R^{(1)}$, $H_1^{(1)}$, and $H_2^{(1)}$ (or, more generally, the solution $U(t)$ to the Schrödinger equation $\mathrm{i}\partial_t U(t) = H_g^{(1)}(t)U(t)$, $U(0) \equiv 1$) can be written as the product of a single-mode rotation, displacement and squeeze operator, and an overall phase factor (see subsection 2.3 and appendix A). Since these Hermitian forms, or time-dependent Hamiltonians, describe all physical processes that produce single-mode GPS (proved in subsection 2.2, by considering the most general single-mode Gaussian wave functions), the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ of eq. (1.16), with r and φ defined over the ranges (1.12b), represents the most general (normalized) single-mode GPS.

Single-mode squeezed states were introduced independently by Stoler [12] ("minimum-uncertainty packets") and Lu [13] ("new coherent states"). They have been discussed in detail by Yuen [16] in the context of quantum optics under the name "two-photon coherent states" or "TCS". Their properties and possible application to back-action evading techniques [4] for gravitational-wave detection were first considered by Hollenhorst [17], who coined the adjective "squeezed". For more recent discussions see, e.g., refs. [5] and [6]. Single-mode squeezed states, described in the context of "generalizations" of coherent states, have also been discussed from group theoretical viewpoints by Barut and Girardello [14], Perelomov [15], Milburn [33], and others [34]. The properties of single-mode squeezed states are summarized briefly here and below in subsections 2.1.5 and 2.2.

Recall that the total noise of a single-mode GPS is the sum of the variances of the real and imaginary parts (mean-square uncertainty) of the annihilation operator a, or, equivalently, of $e^{i\delta}a$, where δ is any real number. The total noise of a single-mode coherent state is equal to $\frac{1}{2}$, the smallest value allowed by quantum mechanics (the half quantum of zero-point noise). This implies that, for all choices of δ, the two variances are equal to each other, and their product is equal to the minimum value allowed by quantum mechanics. Contrast this with single-mode squeezed states. For certain ranges of the squeeze angle φ (or, equivalently, for those conjugate observables defined by certain ranges of δ), one of the variances is smaller than it would be in a coherent state. The other variance is greater than it would be in a coherent state, since the total noise of a SMSS is greater, but this does not alter the potential practical advantages offered by the reduced uncertainty in the one observable. These advantages are the impetus for the current experimental effort to produce squeezed states [35–37]; applications have been proposed in low-noise optical communications [1, 2] and high-precision interferometric experiments [38–40], for example. For a single-mode squeezed state the variance of the real part of $e^{-i\varphi}a$ is minimized and is a factor e^{-2r} smaller than its coherent-state value, while the variance of the imaginary part of $e^{-i\varphi}a$ is maximized and is a factor e^{2r} larger than its coherent-state value. The product of these variances is equal to its minimum allowed value, as in a coherent state.

The important parameter of a squeezed state is its squeeze factor r, not its squeeze angle φ. There are a number of equivalent ways to see this. First, as conjugate observables the real and imaginary parts of a deserve no special status relative to the real and imaginary parts of $e^{i\delta}a$. In actual experiments one would tune the apparatus to respond to whichever observable has the smallest uncertainty. Second, as the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ evolves freely, its squeeze angle changes, but its squeeze factor r does not. The uncertainties oscillate between the conjugate observables (as does the energy between potential and kinetic), but the total noise, which depends only on r, is constant. Even if an oscillator in a SMSS is acted on by a classical force (i.e., multiplied by a displacement operator), its squeeze factor remains constant, and only its squeeze angle and eigenvalue (as well as its complex amplitude $\langle a\rangle$) change. If, however, an oscillator in a SMSS is subjected to a new degenerate two-photon interaction $[H_2^{(1)}(t)]$ – i.e., multiplied by another single-mode squeeze operator – it will go into another SMSS, with different squeeze factor, squeeze angle, and eigenvalue.

1.7. Two-mode Gaussian pure states

For two modes, there are three interaction Hamiltonians in $H_2^{(2)}(t)$ that can produce a GPS whose total noise is greater than that of a coherent state. Two of these are the degenerate two-photon interaction Hamiltonians $H_2^{(1)}(t)$ of eq. (1.8), one for each mode. The third is the nondegenerate two-photon interaction Hamiltonian $H_{2+-}(t)$ defined in eq. (1.7). The most general kind of (normalized) two-mode GPS is produced when two oscillators, each in a coherent state, are exposed to all three of these quadratic interaction Hamiltonians. Formally, this state, symbolized by the state vector $|\boldsymbol{\mu}_g\rangle$ (or $|\mu_{g+}, \mu_{g-}\rangle$), is related to a two-mode coherent state by a product of the three squeeze operators:

$$|\boldsymbol{\mu}_g\rangle \equiv S_{1+}(r_+, \varphi_+)S_{1-}(r_-, \varphi_-)S(r, \varphi)|\mu_g\rangle_{\mathrm{coh}} \equiv \mathbf{S}|\mu_g\rangle_{\mathrm{coh}}\,. \tag{1.18}$$

It is an eigenstate of the transformed annihilation operators $g_\pm \equiv \mathbf{S}a_\pm\mathbf{S}^\dagger$, with complex eigenvalues $\mu_{g\pm}$. The order of the three squeeze operators in the operator $\mathbf{S}$ of eq. (1.18) has been chosen for convenience only. All states unitarily related to the GPS $|\boldsymbol{\mu}_g\rangle$ of eq. (1.18) by a product of rotation, displacement, mixing, and squeeze operators can be expressed as another two-mode GPS $|\boldsymbol{\mu}'_g\rangle$ (multiplied by an unobservable phase factor), with new values for $r_\pm$, r, $\varphi_\pm$, φ, μ_{g+}, and μ_{g-}. Any unitary operator $U \equiv \exp[-iH_g^{(2)}t]$ whose generator $H_g^{(2)}$ is a linear combination of the Hermitian forms $H_R^{(2)}$, $H_1^{(2)}$, and $H_2^{(2)}$ can be written as the product (in any order) of two single-mode rotation and displacement operators, a mixing operator, an operator like S, and an overall phase factor (see subsection 3.3 and appendix A). Since these Hermitian forms, or time-dependent Hamiltonians, describe all physical processes that produce two-mode GPS (proved in subsection 3.3, by considering the most general two-mode Gaussian wave function), the state $|\boldsymbol{\mu}_g\rangle$ defined by eq. (1.18) represents the most general normalized two-mode GPS.

When two oscillators, each in a coherent state, are subjected only to degenerate two-photon interactions $[H_2^{(1)}(t)]$, the resulting two-mode state is simply a tensor product of two single-mode squeezed states (no correlations between the modes). If, however, they are subjected only to a nondegenerate two-photon interaction $[H_{2+-}(t)]$, the resulting state is called a "two-mode squeezed state" [18–21] (TMSS). Formally, a TMSS, symbolized by the state vector $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$, is defined as that state unitarily related to the two-mode coherent state $|\boldsymbol{\mu}_\alpha\rangle_{\rm coh}$ by the two-mode squeeze operator,

$$|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)} \equiv |\mu_{\alpha+}, \mu_{\alpha-}\rangle_{(r,\varphi)} \equiv S(r,\varphi)|\boldsymbol{\mu}_\alpha\rangle_{\rm coh}\,. \tag{1.19}$$

The TMSS $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$ is an eigenstate of the "two-mode squeezed annihilation operators" [18–20]

$$\alpha_\pm(r,\varphi) \equiv S(r,\varphi)a_\pm S^\dagger(r,\varphi)\,. \tag{1.20}$$

with complex eigenvalues $\mu_{\alpha\pm}$. The properties and importance of two-mode squeezed states in the context of quantum optics are the subject of a recent series of papers by Caves and me [19, 20]. They are discussed further in subsection 1.8 below and in section 3.

How might one produce light in the general two-mode GPS $|\boldsymbol{\mu}_g\rangle$ of eq. (1.18)? The simplest answer for purpose of illustration is to use three parametric amplifiers. One could first shine coherent-state light on a nonlinear medium pumped at frequency 2Ω that is phase-matched out (at least) to frequencies $\Omega \pm \varepsilon$; this produces the desired "broadband squeezing". Next, this light would be used as input to another nonlinear medium(s) pumped at frequencies $2(\Omega+\varepsilon)$ and $2(\Omega-\varepsilon)$, with negligible phase matching between the two frequencies. This produces the desired additional degenerate squeezing at the two sideband frequencies.

1.8. Two-mode squeezed states

The two-mode squeezed states (1.19) are the natural two-mode analogs of single-mode squeezed states [eq. (1.16)]. Formally, this is because they are unitarily related to two-mode coherent states in the same functional way that single-mode squeezed states are related to single-mode coherent states. This is a consequence of the fact that the three operators a_+a_-, $a_-^\dagger a_+^\dagger$, and $(a_+a_+^\dagger + a_-a_-^\dagger)_{\rm sym}$ have the same commutator algebra [that of the noncompact, pseudo-unitary Lie group SU(1, 1)] as the operators $\frac{1}{2}a^2$, $\frac{1}{2}a^{\dagger 2}$, and $(aa^\dagger)_{\rm sym}$ (see subsections 2.3, 3.1.5b,c, and 3.3) [15, 41]. Physically, a two-mode squeezed state can be produced in a parametric amplifier by using a single pump whose photons have energy $\Omega_+ + \Omega_-$, just as a single-mode squeezed state can be produced in the degenerate limit of a parametric amplifier by using a single pump with photons of energy 2Ω. Thus, a two-mode squeezed state is produced automatically in a degenerate parametric amplifier if one looks slightly away from degeneracy. In contrast, production of the general two-mode GPS (1.18) would require three separate parametric amplifiers – i.e., three different pumps, with photon energies $2\Omega_+$, $2\Omega_-$, and $\Omega_+ + \Omega_-$.

Like a single-mode squeezed state, a two-mode squeezed state is a state in which the variance of one of two conjugate observables is smaller than it would be in a coherent state. For a single-mode squeezed state the natural conjugate observables are the real and imaginary parts of a (or $e^{i\delta}a$). But what are they for two-mode squeezed states? Analyses of optical heterodyning [1, 42], together with the properties of two-mode squeezed states, indicate that natural choices for these observables are the quadrature-phase operators E_1 and E_2 of the electric field E (or similarly defined quantities if the oscillators are not modes of the electromagnetic field). The following qualitative remarks give a general idea of the nature and significance of the quadrature phases. For further discussion, the reader is referred to appendix D and refs. [1, 2, 18–21] and [42–46].

In optical heterodyning an input electric field $E(t) \propto E_1(t)\cos\Omega t + E_2(t)\sin\Omega t$, composed of upper and lower sidebands of a carrier frequency Ω, is combined at a beam splitter with a strong local-oscillator (LO) field at the carrier frequency; for illustration here let the LO field have time dependence $\cos\Omega t$. One or both of the beam-splitter output ports is then monitored with a photodetector(s). The preferred method monitors both output ports, with identical photodetectors, and coherently subtracts their outputs ("balanced heterodyning"; see refs. [45, 46] and [42]). The relative strength of the LO field guarantees that (i) the dominant contribution to the output intensity is proportional to the mean field of the in-phase quadrature phase, $\langle E_1\rangle$ in this example, and (ii) the dominant contribution to the noise in the output intensity is proportional to the noise in (variance of) that quadrature phase. A reduction in the noise in one quadrature phase relative to its coherent-state value (squeezing) is therefore manifested in heterodyning as an improved signal-to-noise ratio.

The upper and lower sidebands of the input field consist of modes with frequencies $\Omega+\varepsilon$ and $\Omega-\varepsilon$, respectively, where the "modulation frequencies" ε take on all positive values in some bandwidth $\Delta\varepsilon$ ($0\le\Delta\varepsilon\ll\Omega$). The quadrature phases have no time dependence at the carrier frequency Ω; they carry only the time dependences at frequencies ε. Thus, the signal observed at the output of a heterodyne detector is a modulation of the local-oscillator mean field, with modulation frequencies ε. Typically, one filters this signal to pick out the contribution from a single modulation frequency ε, i.e., from one pair of modes, with frequencies $\Omega\pm\varepsilon$. The noise properties of this filtered output modulation signal thus reflect the noise properties of a two-mode state.

When measured in units of energy, the minimum contribution that two modes with frequencies $\Omega\pm\varepsilon$ can make to the time-averaged noise in (variance of) the electric field is the sum of their individual minimum total noise energies, $\frac{1}{2}(\Omega+\varepsilon)+\frac{1}{2}(\Omega-\varepsilon)=\Omega$. This minimum is realized only if both modes are in coherent states; together they contribute a zero-point noise energy of $\frac{1}{2}\Omega$ to each quadrature phase. As shown below, the minimum contribution that two modes of frequencies $\Omega\pm\varepsilon$ can make to the time-averaged noise in either quadrature phase is $\frac{1}{2}\varepsilon$, much smaller than that realized by a two-mode coherent state [eqs. (1.22), appendix D, or refs. [18–20]]. For the time-averaged noise in one quadrature phase to be smaller than its coherent-state value of $\frac{1}{2}\Omega$, the two modes must be specially correlated with each other, in the way produced by a nondegenerate two-photon interaction like (1.7); that is, the two modes must be in a state whose unitary relation to a two-mode coherent state includes a two-mode squeeze operator $S(r,\varphi)$. The reduction is greatest when the two modes are in a two-mode squeezed state (see appendix D).

The obvious advantage squeezing offers is that one can transmit a signal at frequencies $\Omega\pm\varepsilon$ as amplitude or phase modulation of a carrier wave at frequency Ω (modulation frequency $0\le\varepsilon\ll\Omega$), and have a time-averaged (zero-point) noise associated with that signal that is much smaller than the zero-point noises $\frac{1}{2}(\Omega\pm\varepsilon)$ that would accompany the same signal if it were sent directly at the frequencies $\Omega\pm\varepsilon$. The quadrature-phase zero-point noise energy $\frac{1}{2}\varepsilon$ is very small, and with real photodetectors essentially unobservable. That it is in principle nonzero (for nonzero ε), however, is consistent with what one might expect physically; it says that the zero-point noise of $\frac{1}{2}\varepsilon$ associated with any signal transmitted directly at frequency ε cannot be made to vanish by "disguising" the signal as amplitude or phase modulation of a carrier wave at frequency $\Omega\gg\varepsilon$.

The properties of two-mode GPS can be described in terms of the annihilation and creation operators ($a_\pm$ and $a_\pm{}^\dagger$) of the two modes; this is the approach taken in section 3 of this paper. Alternatively, they can be described in terms of the "quadrature-phase amplitudes" α_1 and α_2 for the two modes [18–21]. The latter are dimensionless complex operators, proportional to the Fourier components at frequency ε of the electric field quadrature phases E_1 and E_2 (see appendix D). They are defined as the following linear combinations of a_+ and $a_-{}^\dagger$:

$$\alpha_1 \equiv (2\Omega)^{-1/2}[(\Omega+\varepsilon)^{1/2}a_+ + (\Omega-\varepsilon)^{1/2}a_-^\dagger]\,, \tag{1.21a}$$

$$\alpha_2 \equiv (2\Omega)^{-1/2}[-\mathrm{i}(\Omega+\varepsilon)^{1/2}a_+ + \mathrm{i}(\Omega-\varepsilon)^{1/2}a_-^\dagger]\,. \tag{1.21b}$$

In units of energy, the time-averaged noise in (variance of) the quadrature phase E_j ($j = 1$ or 2) at frequency ε is proportional to the product of Ω and the mean-square uncertainty in α_j (the sum of the variances of its real and imaginary parts). The quadrature-phase zero-point noise energy $\frac{1}{2}\varepsilon$ is a consequence of the commutation relations of the quadrature-phase amplitudes:

$$[\alpha_1, \alpha_1^\dagger] = [\alpha_2, \alpha_2^\dagger] = \varepsilon/\Omega\,; \qquad [\alpha_1, \alpha_2^\dagger] = [\alpha_1^\dagger, \alpha_2] = \mathrm{i}\,; \qquad [\alpha_1, \alpha_2] = 0\,. \tag{1.22}$$

These imply that the minimum mean-square uncertainty in α_j is $\frac{1}{2}\varepsilon/\Omega$; it is realized in a special kind of two-mode squeezed state (a "squashed state", for which $c = \frac{1}{2}\cosh^{-1}\Omega/\varepsilon$ [19]). They also imply that the minimum value for the product of the total noises in α_1 and α_2 is $\frac{1}{2}$; it is realized by two-mode coherent states ([19] and [20]).

For a two-mode squeezed state the only nonvanishing noise moments of α_1 and α_2 (moments of $\Delta\alpha_1 \equiv \alpha_1 - \langle\alpha_1\rangle$ and $\Delta\alpha_2 \equiv \alpha_2 - \langle\alpha_2\rangle$) are those that contain equal numbers of quadrature-phase amplitudes and their Hermitian conjugates, e.g., $\langle\Delta\alpha_1\Delta\alpha_2^\dagger\rangle$, $\langle\Delta\alpha_1^\dagger\Delta\alpha_1\rangle$, etc. This has two important consequences: First, all time-dependent noise moments of the quadrature phases E_1 and E_2 vanish. Fields with this property are said to have "time-stationary quadrature-phase" (TSQP) noise [18–21]. Second, the description of the properties of two-mode squeezed states is formally identical to that for single-mode squeezed states (see the discussions in subsections 3.1.5b,c and 3.3).

States that do not exhibit TSQP noise – e.g., products of two single-mode squeezed states, or the general two-mode GPS $|\boldsymbol{\mu}_g\rangle$ – always have a mean-square uncertainty in α_j (hence a time-averaged noise in E_j) that is greater than the coherent-state value. However, such states can exhibit squeezing of another sort, i.e., squeezing in a pair of conjugate observables other than the quadrature phases E_1 and E_2 [56]. The operator for the modulation signal at the output of a heterodyne detector [the frequency ε component of $E_j(t)$, say] has the form $\alpha_{j1}\cos\varepsilon t + \alpha_{j2}\sin\varepsilon t$, where α_{j1} (α_{j2}) is $2^{1/2}$ times the real (imaginary) part of the quadrature-phase amplitude α_j (see appendix D). For a two-mode squeezed state the vanishing of the noise moments $\langle(\Delta\alpha_j)^2\rangle$ means that the variances of the observables α_{j1} and α_{j2} are equal. But for other two-mode GPS these variances can be unequal, with one smaller than its coherent-state value. Such states exhibit squeezing in α_{j1} and α_{j2} – i.e., in the quadrature phases of the (monochromatic) modulation signal. There is in principle no limit to this kind of squeezing: the variance of the observable α_{j1} or α_{j2} can be arbitrarily small. For a two-mode coherent state, the variances of α_{j1} and α_{j2} (in energy units) are both $\frac{1}{2}\Omega$; for a two-mode squeezed state, they are again equal and hence can be made only as small as $\frac{1}{2}\varepsilon$ [see eq. (D.7a) below]. But for a product of two single-mode squeezed states with $\varphi_+ = \varphi_-$ and $r_+ - r_- \simeq \varepsilon/\Omega$ [eq. (D.8) below], the variance of α_{j1} or α_{j2} can be a factor $(1-\varepsilon^2/\Omega^2)^{1/2}\exp[-(r_+ + r_-)]$ smaller than $\frac{1}{2}\Omega$, i.e., it can be arbitrarily small. And, for a two-mode GPS like $|\boldsymbol{\mu}_g\rangle$ [eq. (1.18)] with $\varphi_+ = \varphi_- = \varphi$ and $r_+ - r_- \simeq \varepsilon/\Omega$, the variance of either α_{j1} or α_{j2} can be still smaller – a factor $(1-\varepsilon^2/\Omega^2)^{1/2}\exp[-(r_+ + r_-)]\exp(-2r)$ smaller than $\frac{1}{2}\Omega$. This kind of squeezing can be detected by mixing the modulation signal at the output of a heterodyne detector with a wave $\cos(\varepsilon t + \delta)$. The resulting zero-frequency output is proportional to $\langle\alpha_{j1}\rangle$ (if $\delta = 0$) or $\langle\alpha_{j2}\rangle$ (if $\delta = \frac{1}{2}\pi$). (For more details see appendix D.).

1.9. Outline of this paper

Section 2 of this paper is a review of single-mode Gaussian pure states. Subsection 2.1 looks at the unitary operators associated with single-mode GPS and reviews some of the properties of coherent states and single-mode squeezed states. Subsection 2.2 considers the most general single-mode Gaussian wave function and from it shows that the most general single-mode GPS is a single-mode squeezed state. Subsection 2.3 uses a two-component vector notation to provide a compact and powerful way to express the properties of single-mode GPS and their associated unitary operators. Section 3 is a detailed discussion of two-mode Gaussian pure states which parallels closely in structure but is necessarily more complicated than that of section 2.

Some useful details are relegated to appendices. Appendix A outlines the procedure and gives supporting details for writing the unitary evolution operator associated with the most general (time-dependent) linear combination of interaction Hamiltonians that can produce single-mode and two-mode GPS as a product of squeeze, rotation, mixing, and displacement operators. Appendix B calculates the (amplitude-independent) phase factor in the coordinate-space wave functions for single-mode and two-mode squeezed states, and suggests its form for the most general two-mode GPS. Appendix C elaborates on a point made in subsection 3.2 concerning the criterion for two complex operators, defined as linear combinations of creation and annihilation operators, to have a complete (or overcomplete) set of simultaneous, normalizable eigenstates. Appendix D gives some supporting details for the discussion in subsection 1.8 of the kinds of squeezing exhibited by general two-mode GPS.

2. Single-mode Gaussian pure states

2.1. Introduction and review

This section, composed of five subsections, serves both as an introduction to and a review of single-mode GPS. Subsection 2.1.1 defines dimensionless position and momentum variables, and the second-order "noise moments" that characterize single-mode GPS. Subsections 2.2.2–2.2.4 look closely at the unitary operators – rotation, displacement, and squeeze operators – that relate single-mode GPS to the vacuum state. Subsection 2.2.5 derives the second-order noise moments of a single-mode squeezed state, and discusses two special kinds of single-mode states: single-mode "minimum-uncertainty states" (MUS), and single-mode states with "time-stationary" (TS), or "random-phase", noise.

2.1.1. Notation and definitions

The quantum mechanical operators naturally associated with a harmonic oscillator are the Schrödinger-picture (SP) annihilation operator a and its adjoint $a^\dagger$, the creation operator. Equivalent operators are the dimensionless position and momentum $\hat{x}$ and $\hat{p}$; these are Hermitian operators, constant in the SP and related to a and $a^\dagger$ by

$$\hat{x} \equiv 2^{-1/2}(a + a^\dagger)\,, \qquad \hat{p} \equiv 2^{-1/2}(-\mathrm{i}a + \mathrm{i}a^\dagger)\,; \tag{2.1.1a}$$

$$a \equiv 2^{-1/2}(\hat{x} + \mathrm{i}\hat{p})\,. \tag{2.1.1b}$$

The position and momentum are equal to $2^{1/2}$ times the real and imaginary parts of a, respectively. The creation and annihilation operators and the dimensionless position and momentum obey the standard commutation relations:

$$[a, a^\dagger] = 1\,, \qquad [\hat{x}, \hat{p}] = \mathrm{i}\,. \tag{2.1.2}$$

The complex amplitude of a single-mode state, always denoted in this paper by the symbol μ, is the expectation value of a; it is related to the mean position and momentum x_0 and p_0 by

$$\mu \equiv \langle a\rangle = 2^{-1/2}(\langle\hat{x}\rangle + \mathrm{i}\langle\hat{p}\rangle) \equiv 2^{-1/2}(x_0 + \mathrm{i}p_0)\,. \tag{2.1.3}$$

The noise moments of the operators a and $a^\dagger$ or $\hat{x}$ and $\hat{p}$ provide a useful way to characterize states associated with harmonic oscillators. Noise moments of any operator B are moments of $\Delta B \equiv B - \langle B\rangle$, the operator minus its mean. Note that an operator ΔB is defined only with reference to a particular state, which defines $\langle B\rangle$. All noise moments of a and $a^\dagger$ (or $\hat{x}$ and $\hat{p}$) for Gaussian states are expressible in terms of the second-order noise moments. There are three real second-order noise moments of the annihilation operator a. Two of these make up the complex number

$$\langle(\Delta a)^2\rangle \equiv \langle a^2\rangle - \langle a\rangle^2 = \langle(\Delta a^\dagger)^2\rangle^*\,, \tag{2.1.4a}$$

and the third is the positive real number

$$\langle|\Delta a|^2\rangle \equiv \langle\Delta a\,\Delta a^\dagger\rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}\langle\Delta a\,\Delta a^\dagger + \Delta a^\dagger\,\Delta a\rangle\,, \tag{2.1.4b}$$

the mean-square uncertainty in a (the subscript "sym" denotes a symmetrized product). These second-order noise moments of a are related to the three real second-order noise moments of $\hat{x}$ and $\hat{p}$ by

$$\langle(\Delta a)^2\rangle = \tfrac{1}{2}(\langle(\Delta\hat{x})^2\rangle - \langle(\Delta\hat{p})^2\rangle) + \mathrm{i}\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}}\,, \tag{2.1.5a}$$

$$\langle|\Delta a|^2\rangle = \tfrac{1}{2}(\langle(\Delta\hat{x})^2\rangle + \langle(\Delta\hat{p})^2\rangle) \tag{2.1.5b}$$

[eqs. (2.1.1)]. The total (second-moment) noise of a single-mode GPS is the mean-square uncertainty $\langle|\Delta a|^2\rangle$, the sum of the variances (squared uncertainties) of the real and imaginary parts of a.

The commutation relations (2.1.2) enforce the following lower limits on the product and sum of the variances of $\hat{x}$ and $\hat{p}$ [47]:

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \tfrac{1}{4} + \langle\Delta\hat{x}\,\Delta\hat{p}\rangle^2_{\mathrm{sym}} \geq \tfrac{1}{4}\,, \tag{2.1.6a}$$

$$\tfrac{1}{4}(\langle(\Delta\hat{x})^2\rangle + \langle(\Delta\hat{p})^2\rangle)^2 = \langle|\Delta a|^2\rangle^2 \geq \tfrac{1}{4} + |\langle(\Delta a)^2\rangle|^2 \geq \tfrac{1}{4}\,. \tag{2.1.6b}$$

Equalities hold in the first of each of these inequalities if and only if the state is an eigenstate of a certain kind of linear combination of $\hat{x}$ and $\hat{p}$ (or a and $a^\dagger$) – i.e., if and only if the state is a Gaussian pure state (see subsection 2.2). Hence for a single-mode GPS only two of the three real second-order noise moments are independent; i.e., there are two independent real numbers associated with the second-order noise moments.

2.1.2. Single-mode rotation operator

Consider now the single-mode rotation operator $R(\theta)$, defined by

$$R(\theta) \equiv \exp(-\mathrm{i}\theta a^\dagger a) = \exp(\tfrac{1}{2}\mathrm{i}\theta)\exp[-\tfrac{1}{2}\mathrm{i}\theta(\hat{x}^2 + \hat{p}^2)] \tag{2.1.7a}$$

[eq. (1.9)]. It satisfies

$$R^{-1}(\theta) = R^{\dagger}(\theta) = R(-\theta)\,. \tag{2.1.7b}$$

For an oscillator characterized by frequency Ω, $R(\Omega t)$ is the evolution operator associated with the free Hamiltonian $H_0^{(1)}$,

$$H_0^{(1)} \equiv \Omega a^{\dagger} a = \tfrac{1}{2}\Omega(\hat{x}^2 + \hat{p}^2 - 1)\,, \tag{2.1.8a}$$

$$\exp(-\mathrm{i}H_0^{(1)}t) = R(\Omega t)\,. \tag{2.1.8b}$$

The rotation operator acting on any number eigenstate $|n\rangle$ simply multiplies it by the phase factor $\mathrm{e}^{-\mathrm{i}n\theta}$ [eqs. (1.2)]; in particular, it leaves the vacuum state unchanged:

$$R(\theta)|0\rangle = |0\rangle\,. \tag{2.1.9}$$

The rotation operator unitarily transforms the annihilation operator a into $\mathrm{e}^{\mathrm{i}\theta}a$ – i.e., it rotates $\hat{x}$ and $\hat{p}$ into each other:

$$R(\theta)aR^{\dagger}(\theta) = \mathrm{e}^{\mathrm{i}\theta}a \equiv a(\theta)\,, \tag{2.1.10a}$$

$$R(\theta)\hat{x}R^{\dagger}(\theta) = \hat{x}\cos\theta - \hat{p}\sin\theta \equiv \hat{x}(\theta)\,, \tag{2.1.10b}$$

$$R(\theta)\hat{p}R^{\dagger}(\theta) = \hat{x}\sin\theta + \hat{p}\cos\theta \equiv \hat{p}(\theta)\,. \tag{2.1.10c}$$

The unitarity of $R(\theta)$ ensures that $\hat{x}(\theta)$ and $\hat{p}(\theta)$ are conjugate observables, $[\hat{x}(\theta), \hat{p}(\theta)] = \mathrm{i}$. The transformation (2.1.10a) shows that an eigenstate of a remains an eigenstate of a when operated on by a rotation operator, i.e., as it evolves freely. The rotation operator clearly preserves the total number of photons in the mode,

$$R^{\dagger}(\theta)a^{\dagger}aR(\theta) = a^{\dagger}a \tag{2.1.11}$$

(hence also the total energy). The effect of the rotation operator is merely to transfer energy between kinetic ($\hat{p}^2$) and potential ($\hat{x}^2$). It therefore also preserves the total noise,

$$\langle R^{\dagger}(\theta)|\Delta a|^2 R(\theta)\rangle = \langle|\Delta a|^2\rangle\,, \tag{2.1.12a}$$

its effect on a state being merely to redistribute the noise between the position and momentum variables,

$$\langle R^{\dagger}(\theta)(\Delta a)^2 R(\theta)\rangle = \langle[\Delta a(-\theta)]^2\rangle = \mathrm{e}^{-2\mathrm{i}\theta}\langle(\Delta a)^2\rangle\,. \tag{2.1.12b}$$

Note in eqs. (2.1.12) that the operator Δa on the left-hand side of the equations is $\Delta a \equiv a - \langle R^{\dagger}(\theta)aR(\theta)\rangle$, whereas on the right-hand side it is $\Delta a \equiv a - \langle a\rangle$. A similar remark holds throughout this paper wherever the moments or noise moments of operators in a state $|\Psi\rangle$ are compared with those in a state $U|\Psi\rangle$.

Finally, note that the simple form of $R(\theta)$ implies that the product of an arbitrary number of rotation operators can be expressed trivially as a single rotation operator, using the rule

$$R(\theta)R(\theta') = R(\theta + \theta')\,. \tag{2.1.13}$$

2.1.3. Single-mode displacement operator

The single-mode displacement operator [10, 11] is defined by

$$D(a, \mu) \equiv \exp(\mu a^\dagger - \mu^* a) = \exp[\mathrm{i}(p_0\hat{x} - x_0\hat{p})] = \exp(-\tfrac{1}{2}\mathrm{i}p_0x_0)\exp(\mathrm{i}p_0\hat{x})\exp(-\mathrm{i}x_0\hat{p}) \tag{2.1.14}$$

[eq. (1.11)]. It satisfies the following equalities:

$$D^{-1}(a, \mu) = D^\dagger(a, \mu) = D(a, -\mu) = D(-a, \mu)\,. \tag{2.1.15}$$

Properties of $D(a, \mu)$ are discussed in refs. [10, 11, 31, 20]. Most important is the way it unitarily transforms the annihilation operator:

$$D(a, \mu)aD^\dagger(a, \mu) = a - \mu\,. \tag{2.1.16}$$

The additive nature of this transformation implies that when the displacement operator acts on a state it changes all moments of a and $a^\dagger$ (e.g., the complex amplitude $\langle a\rangle$, and the photon number $\langle a^\dagger a\rangle$). However, since the transformation merely adds a complex number to a, the noise moments of a and $a^\dagger$ are left unchanged. Thus, when the displacement operator acts on a state, it displaces the wave function, but does not modify its shape. In particular, an eigenstate of a remains an eigenstate of a when operated on by a displacement operator. Hence the single-mode coherent state $|\mu\rangle_{\mathrm{coh}}$, defined as $D(a, \mu)$ acting on the vacuum state [eq. (1.14)], is an eigenstate of a with eigenvalue μ.

Two other properties of the displacement operator are useful here. First, it is unitarily transformed by the rotation operator in the following way:

$$R(\theta)D(a, \mu)R^\dagger(\theta) = D[a(\theta), \mu] = D[a, \mu(-\theta)]\,, \tag{2.1.17a}$$

$$\mu(\theta) \equiv e^{\mathrm{i}\theta}\mu \tag{2.1.17b}$$

[eqs. (2.1.10), (2.1.14)]. This transformation shows that the form of the displacement operator is invariant under a unitary transformation of a generated by the rotation operator:

$$D(a, \mu) = D[a(\theta), \mu(\theta)]\,. \tag{2.1.18}$$

It also shows that $D(a, \mu)$ does not commute with the rotation operator and hence does not preserve photon number. Second, the product of two displacement operators is another displacement operator, multiplied by a phase factor:

$$D(a, \mu')D(a, \mu) = \exp[\mathrm{i}\,\mathrm{Im}(\mu\mu'^*)]D(a, \mu + \mu')\,. \tag{2.1.19}$$

These properties, like the transformations (2.1.10a) and (2.1.16), show that any eigenstate of a remains an eigenstate of a when displaced and/or allowed to evolve freely. For example, as a coherent state [eq. (1.14)] evolves freely, it changes in the following way:

$$R(\Omega t)|\mu\rangle_{\mathrm{coh}} = |\mu(-\Omega t)\rangle_{\mathrm{coh}} \equiv |e^{-\mathrm{i}\Omega t}\mu\rangle_{\mathrm{coh}}\,. \tag{2.1.20}$$

All single-mode states that are eigenstates of a are unitarily related to the vacuum state by products of rotation and displacement operators. Conversely, all such states are eigenstates of a. These states comprise the entire class of single-mode states whose total noise is equal to that of the vacuum state. The special properties of the rotation operator – that it preserves the total number of photons, that it preserves the total noise, and that it preserves coherent states – are a consequence of one essential property: the unitary transformation it induces on a merely multiplies a by a phase factor; i.e., it never mixes a with $a^\dagger$. To find unitary operators that do not conserve the total noise and that generate new states from coherent states (states with a total noise greater than that of the vacuum state), one must consider operators – single-mode squeeze operators – that mix a with $a^\dagger$.

2.1.4. Single-mode squeeze operator

The single-mode squeeze operator [12, 13, 17] is defined by

$$S_1(r, \varphi) \equiv \exp[\tfrac{1}{2}r(e^{-2i\varphi}a^2 - e^{2i\varphi}a^{\dagger 2})], \tag{2.1.21a}$$

$$0 \le r < \infty, \qquad -\tfrac{1}{2}\pi < \varphi \le \tfrac{1}{2}\pi \tag{2.1.21b}$$

[eqs. (1.12)]; it is denoted simply by S_1 when lack of reference to a particular r and φ does not lead to confusion. It satisfies the following equalities:

$$S_1^{-1}(r, \varphi) = S_1^\dagger(r, \varphi) = S_1(-r, \varphi) = S_1(r, \varphi + \tfrac{1}{2}\pi). \tag{2.1.22}$$

Properties of $S_1(r, \varphi)$ are discussed in refs. [17, 19] and [20]. Most important is the way it unitarily transforms the annihilation operator:

$$S_1(r, \varphi) a S_1^\dagger(r, \varphi) = a \cosh r + a^\dagger e^{2i\varphi} \sinh r \equiv \alpha(r, \varphi) \tag{2.1.23a}$$

[eq. (1.17)]. A state unitarily related to an eigenstate of a by a single-mode squeeze operator is an eigenstate of the single-mode squeezed annihilation operator $\alpha(r, \varphi)$ (denoted simply by α when lack of reference to r and φ does not lead to confusion). Inverting eq. (2.1.23a) gives a in terms of α and $\alpha^\dagger$:

$$a = S_1^\dagger(r, \varphi)\alpha(r, \varphi)S_1(r, \varphi) = \alpha \cosh r - \alpha^\dagger e^{2i\varphi} \sinh r. \tag{2.1.23b}$$

The unitarity of S_1 ensures that $[\alpha, \alpha^\dagger] = [a, a^\dagger] = 1$.

The transformation (2.1.23a) tells one that when the squeeze operator acts on a state it changes the noise moments of a and $a^\dagger$. That is, it modifies the shape of the wave function (and, if the mean position or momentum are nonzero, displaces it as well). In particular, it preserves neither the total number of photons nor the total noise of a state,

$$\langle S_1^\dagger a^\dagger a S_1\rangle = \sinh^2 r + \cosh 2r\langle a^\dagger a\rangle - \sinh 2r\, \mathrm{Re}(e^{-2i\varphi}\langle a^2\rangle), \tag{2.1.24a}$$

$$\langle S_1^\dagger |\Delta a|^2 S_1\rangle = \cosh 2r\langle |\Delta a|^2\rangle - \sinh 2r\, \mathrm{Re}(e^{-2i\varphi}\langle (\Delta a)^2\rangle). \tag{2.1.24b}$$

Equation (2.1.24b) shows explicitly that any state whose unitary relation to the vacuum state (or to any eigenstate of a) includes a single-mode squeeze operator has a total noise greater than that of the vacuum state.

A few other properties of the single-mode squeeze operator are useful here. First, it is unitarily transformed by the rotation operator in the following way:

$$R(\theta)S_1(r,\varphi)R^\dagger(\theta)=S_1(r,\varphi-\theta) \tag{2.1.25}$$

[eqs. (2.1.10a), (2.1.21)]. That $S_1(r,\varphi)$ does not commute with the rotation operator reveals why it does not preserve photon number [eq. (2.1.24a)].

Second, it unitarily transforms the displacement operator in the following way:

$$S_1^\dagger(r,\varphi)D(a,\mu)S_1(r,\varphi)=D(a,\mu_\alpha)\,, \tag{2.1.26}$$

$$\mu_\alpha \equiv \mu\cosh r+\mu^*\,\mathrm{e}^{2\mathrm{i}\varphi}\sinh r \tag{2.1.27}$$

[eqs. (2.1.14), (2.1.23a)]. This relation reflects the fact that the form of the displacement operator is invariant under unitary transformations of a that are linear in a and $a^\dagger$ (and that do not add to a a constant). Such unitary transformations are generated only by (products of) rotation and single-mode squeeze operators. The invariance under rotations was noted in eq. (2.1.18). The invariance under transformations generated by the single-mode squeeze operator says that

$$D(a,\mu)=D(\alpha,\mu_\alpha)\,. \tag{2.1.28}$$

This equality implies that the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$, defined by eq. (1.16) as the squeeze operator $S_1(r,\varphi)$ acting on the coherent state $|\mu_\alpha\rangle_{\mathrm{coh}}$, can as well be defined as the displacement operator $D(a,\mu)$ acting on the squeezed vacuum:

$$|\mu_\alpha\rangle_{(r,\varphi)}\equiv S_1(r,\varphi)|\mu_\alpha\rangle_{\mathrm{coh}}=D(a,\mu)S_1(r,\varphi)|0\rangle\,. \tag{2.1.29}$$

The complex number μ is equal to $\langle a\rangle$, the state's complex amplitude. It is related to the eigenvalue μ_α by

$$\mu=\mu_\alpha\cosh r-\mu_\alpha^*\,\mathrm{e}^{2\mathrm{i}\varphi}\sinh r \tag{2.1.30}$$

[eq. (2.1.23b)]. With this definition one can easily verify the statement made in the Introduction: any state unitarily related to the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ by a product of rotation and displacement operators is equal to another SMSS (multiplied by an unobservable overall phase factor) with the same squeeze factor r, but with different squeeze angle and eigenvalue. For example,

$$\begin{aligned} R(\theta)D(a,\mu')|\mu_\alpha\rangle_{(r,\varphi)} &= \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\mu\mu'^*)}D(a,\bar\mu)S_1(r,\varphi-\theta)|0\rangle \\ &= \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\mu\mu'^*)}|\bar\mu_\alpha\rangle_{(r,\varphi-\theta)}\,, \end{aligned}$$

$$\bar\mu\equiv\mathrm{e}^{-\mathrm{i}\theta}(\mu+\mu') \tag{2.1.31}$$

[eqs. (2.1.17), (2.1.19), (2.1.25)].

Finally, the product of two different single-mode squeeze operators is another single-mode squeeze operator, multiplied by a phase factor and a rotation operator [see eqs. (2.3.16) and appendix B of ref. [20]]. For the case $\varphi=\varphi'$ the relation simplifies to

$$S_1(r,\varphi)S_1(r',\varphi)=S_1(r+r',\varphi)\,. \tag{2.1.32}$$

It is proved in the next section, by considering the most general single-mode Gaussian wave function, that the Hermitian generator $H_g^{(1)}$ of the unitary operator $U_g^{(1)} \equiv \exp[-iH_g^{(1)}t]$ that relates a single-mode GPS to the vacuum state is a sum of linear and bilinear combinations of a and $a^\dagger$. In other words, the most general single-mode GPS is produced when a harmonic oscillator in its ground state is exposed to the interaction Hamiltonians $H_R^{(1)}(t)$, $H_1^{(1)}(t)$, and $H_2^{(1)}(t)$ described in the Introduction [eqs. (1.3) and (1.4)]. The unitary operator $U_g^{(1)}$ factors into a product of single-mode displacement, squeeze, and rotation operators (in any order), and an overall phase factor (proved in subsection 2.3 and appendix A). The properties described in this subsection ensure that any product of single-mode rotation, displacement, and squeeze operators can be expressed as the product of a displacement operator and a single-mode squeeze operator (in either order), multiplied on the right by a rotation operator (and an overall phase factor). Since the rotation operator has no effect on the vacuum state, this shows that the most general single-mode GPS can be described as a single-mode squeezed state, defined by eq. (2.1.29).

2.1.5. Single-mode GPS and minimum-uncertainty states (MUS)

The second-order noise moments for a single-mode coherent state follow directly from the fact that it is an eigenstate of the annihilation operator a:

$$\langle(\Delta a)^2\rangle = \langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\text{sym}} = 0\,, \tag{2.1.33a}$$

$$\langle|\Delta a|^2\rangle = \langle(\Delta\hat{x})^2\rangle = \langle(\Delta\hat{p})^2\rangle = \tfrac{1}{2}\,. \tag{2.1.33b}$$

A coherent state has the minimum total noise allowed by quantum mechanics [eq. (2.1.6b)]. The second-order noise moments for the most general single-mode GPS, a single-mode squeezed state, can be obtained from the transformation (2.1.23a) and those for a coherent state, or from eqs. (2.2.32) and (2.2.35) below. For the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ [eq. (2.1.29)] they are

$$\langle(\Delta a)^2\rangle = -\tfrac{1}{2}e^{2i\varphi}\sinh 2r\,, \tag{2.1.34a}$$

$$\langle|\Delta a|^2\rangle = \tfrac{1}{2}\cosh 2r\,; \tag{2.1.34b}$$

$$\langle(\Delta\hat{x})^2\rangle = \tfrac{1}{2}(\cosh 2r - \sinh 2r\cos 2\varphi)\,, \tag{2.1.35a}$$

$$\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{2}(\cosh 2r + \sinh 2r\cos 2\varphi)\,, \tag{2.1.35b}$$

$$\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\text{sym}} = -\tfrac{1}{2}\sinh 2r\sin 2\varphi\,. \tag{2.1.35c}$$

Much of the interest in single-mode GPS has centered around the so-called "minimum-uncertainty states" [12] (MUS) – states that minimize the product of the uncertainties in $\hat{x}$ and $\hat{p}$:

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4}\quad\text{(MUS)} \tag{2.1.36}$$

[eq. (2.1.6a)]. These are (single-mode) GPS that satisfy

$$\operatorname{Im}\langle(\Delta a)^2\rangle = \langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\text{sym}} = 0 \tag{2.1.37}$$

[eqs. (2.1.5)]. The condition (2.1.36) implies that there is only one independent real parameter associated with the second-order noise moments of a single-mode MUS. Comparison with eqs. (2.1.34) or (2.1.35) shows that the set of single-mode MUS consists of single-mode squeezed states with $\varphi = 0$, a set which includes all coherent states ($r = 0$). It is shown below [see eqs. (2.2.5) or (2.2.6)] that a single-mode state is a MUS if and only if it is an eigenstate of a linear combination $\hat{x} + i\gamma_1^{-1}\hat{p}$, $\gamma_1 = \gamma_1^* > 0$. A single-mode squeezed state with $\varphi = 0$ satisfies this condition with $\gamma_1 = e^{2r}$. By comparison, a single-mode state is a Gaussian pure state if and only if it is an eigenstate of a linear combination $\hat{x} + i\gamma^{-1}\hat{p}$, $\mathrm{Re}(\gamma) > 0$ (see section 2.2). All states unitarily related to MUS by a rotation operator $R(\theta)$ are eigenstates of just such a general linear combination:

$$R(\theta)(\hat{x} + i\gamma_1^{-1}\hat{p})R^\dagger(\theta) = \hat{x}(\theta) + i\gamma_1^{-1}\hat{p}(\theta) \propto \hat{x} + i\gamma^{-1}\hat{p}\,, \qquad \mathrm{Re}(\gamma) > 0\,,$$

where $\mathrm{Im}(\gamma)$ is independent of $\mathrm{Re}(\gamma)$ and, in general, nonzero [$\mathrm{Im}(\gamma) = 0$ if and only if $\gamma_1 = 1$, i.e., a rotated coherent state is still a coherent state]. Conversely, all states that are eigenstates of a linear combination $\hat{x} + i\gamma^{-1}\hat{p}$, $\mathrm{Re}(\gamma) > 0$ (i.e., all single-mode GPS) are related to single-mode MUS by rotation operators. Thus, by extending the definition (2.2.36) of single-mode MUS to include all states related to MUS by the rotation operator $R(\theta)$, one obtains all single-mode GPS. Another way to see this is to note that the condition (2.2.37) can always be satisfied for some rotated annihilation operator $a(\theta) \equiv R(\theta)aR^\dagger(\theta) = e^{i\theta}a$ [eq. (2.1.10a)], with θ chosen so that $\mathrm{Im}\langle[\Delta a(\theta)]^2\rangle = 0$. For example, the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ is a MUS for the rotated conjugate variables $\hat{x}(-\varphi)$ and $\hat{p}(-\varphi)$:

$$\langle[\Delta\hat{x}(-\varphi)]^2\rangle = \tfrac{1}{2}e^{-2r}\,, \qquad \langle[\Delta\hat{p}(-\varphi)]^2\rangle = \tfrac{1}{2}e^{2r}\,. \tag{2.1.38a}$$

$$\langle\Delta\hat{x}(-\varphi)\,\Delta\hat{p}(-\varphi)\rangle_{\mathrm{sym}} = 0\,. \tag{2.1.38b}$$

Another important set of single-mode states consists of those that have "random-phase noise", i.e., whose noise moments are invariant under rotations. States with random-phase noise satisfy the condition

$$\langle(\Delta a)^n\rangle = 0\,, \qquad n = 1, 2, 3, \ldots\,. \tag{2.1.39a}$$

For states (pure or mixed) with Gaussian noise statistics, this condition reduces to $\langle(\Delta a)^2\rangle = 0$, which is equivalent to the conditions

$$\langle[\Delta\hat{x}(\theta)]^2\rangle = \langle[\Delta\hat{p}(\theta)]^2\rangle = \langle[\Delta\hat{x}(\theta')]^2\rangle \quad \text{for all } \theta, \theta'\,. \tag{2.1.39b}$$

If all the modes present in an electric field have random-phase noise, the field is said to have "time-stationary" [18–21] (TS) noise, because the condition (2.1.39a) implies that all time-dependent noise moments of the electric field vanish. The intersection between single-mode Gaussian pure states and states with random-phase noise is the set of single-mode coherent states [eqs. (2.1.33)].

2.2. Single-mode Gaussian wave functions

This section begins with the most general single-mode Gaussian coordinate-space wave function, with all parameters arbitrary, subject to normalization. Subsection 2.2.1 explores the relation of the parameters in the wave function to the noise properties of a single-mode GPS, and derives general relations between the different noise moments. Subsection 2.2.2 examines the operators of which single-mode GPS are eigenstates, these being determined by the wave function. It establishes thereby the logical connection between the wave function and the formal definition of a single-mode GPS as a unitary operator acting on the vacuum state. Subsection 2.2.3 derives the unitary operator that relates the most general single-mode GPS to the vacuum state, and shows that the most general single-mode GPS is a single-mode squeezed state. The parameters in the wave function for a single-mode squeezed state are given explicitly in terms of the two real parameters of the single-mode squeeze operator. Subsection 2.2.4 considers the most general single-mode Gaussian momentum-space wave function, and relates its parameters to those of the coordinate-space wave function for a single-mode squeezed state.

2.2.1. *The wave function*

Consider now the coordinate-space wave function for the most general single-mode Gaussian pure state, symbolized here by the state vector $|\mu_g\rangle$. The GPS $|\mu_g\rangle$ is an eigenstate of an operator g, whose general form is discussed below, with complex eigenvalue μ_g. [It will be seen below that g has the form of the single-mode squeezed annihilation operators $\alpha(r, \varphi)$.] The wave function is written in terms of the dimensionless position variable x, the eigenvalue of the Hermitian operator $\hat{x}$. The most general (normalized) single-mode Gaussian coordinate-space wave function has the form

$$\langle x|\mu_g\rangle = N_g \exp(\tfrac{1}{2}\mathrm{i}\delta_x)\exp(\tfrac{1}{2}\mathrm{i}p_0x_0)\exp(\mathrm{i}p_0x)\exp[-\tfrac{1}{2}\gamma(x-x_0)^2]\,. \tag{2.2.1}$$

Here

$$x_0 \equiv \langle\hat{x}\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, x|\langle x|\mu_g\rangle|^2\,, \tag{2.2.2a}$$

$$p_0 \equiv \langle\hat{p}\rangle = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\langle\mu_g|x\rangle\partial_x\langle x|\mu_g\rangle\,, \tag{2.2.2b}$$

are the mean values of the position and momentum, γ is a complex number related to the second-order noise moments of $\hat{x}$ and $\hat{p}$, δ_x is an unobservable phase angle (separated out for reasons discussed below), and N_g is a (real) normalization constant determined by the condition $\langle\mu_g|\mu_g\rangle = 1$. The subscript "$x$" on the phase angle δ_x serves only to distinguish δ_x from the phase angle δ_p which appears in the momentum-space wave function [eqs. (2.2.38)–(2.2.41) below]; δ_x has no dependence on x. Normalizability dictates that

$$\mathrm{Re}(\gamma) \equiv \gamma_1 > 0\,, \tag{2.2.3}$$

and the normalization constant N_g is equal to

$$N_g = (\pi/\gamma_1)^{-1/4}\,. \tag{2.2.4}$$

The most important parameter in the wave function (2.2.1) is the complex number γ. The form of the wave function tells one that the state $|\mu_g\rangle$ is an eigenstate of the linear combination $\hat{x}+\mathrm{i}\gamma^{-1}\hat{p}$, and hence that γ is related to the second-order noise moments of $\hat{x}$ and $\hat{p}$ by

$$\gamma \equiv \gamma_1 + \mathrm{i}\gamma_2 = \frac{-\mathrm{i}(\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}} + \frac{1}{2}\mathrm{i})}{\langle(\Delta\hat{x})^2\rangle} = \frac{-\mathrm{i}\langle(\Delta\hat{p})^2\rangle}{\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}} - \frac{1}{2}\mathrm{i}}\,. \tag{2.2.5a}$$

The real and imaginary parts of γ are therefore equal to

$$\gamma_1 \equiv \mathrm{Re}(\gamma) = \frac{1}{2\langle(\Delta\hat{x})^2\rangle}\,, \qquad \gamma_2 \equiv \mathrm{Im}(\gamma) = \frac{-\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}}}{\langle(\Delta\hat{x})^2\rangle}\,, \tag{2.2.5b}$$

and the absolute square is

$$|\gamma|^2 = \langle(\Delta\hat{p})^2\rangle/\langle(\Delta\hat{x})^2\rangle\,. \tag{2.2.5c}$$

Inverting these expressions gives the second-order noise moments of $\hat{x}$ and $\hat{p}$ in terms of γ:

$$\langle(\Delta\hat{x})^2\rangle = (2\gamma_1)^{-1}\,, \qquad \langle(\Delta\hat{p})^2\rangle = [2\,\mathrm{Re}(\gamma^{-1})]^{-1} = |\gamma|^2(2\gamma_1)^{-1}\,,$$
$$\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}} = -(2\gamma_1)^{-1}\gamma_2\,. \tag{2.2.6}$$

The normalization constant N_g can thus be rewritten as

$$N_g = (\pi/\gamma_1)^{-1/4} = (2\pi\langle(\Delta\hat{x})^2\rangle)^{-1/4}\,. \tag{2.2.7}$$

That the state $|\mu_g\rangle$ is an eigenstate of $\hat{x}+\mathrm{i}\gamma^{-1}\hat{p}$ means that it is also an eigenstate of the operator $a+(\gamma+1)^{-1}(\gamma-1)a^\dagger$. The second-order noise moments of a and $a^\dagger$ are therefore more naturally expressed in terms of the complex number

$$\Gamma \equiv \frac{\gamma-1}{\gamma+1} = \frac{-\langle(\Delta a)^2\rangle}{\langle|\Delta a|^2\rangle + \frac{1}{2}} = \frac{\langle|\Delta a|^2\rangle - \frac{1}{2}}{-\langle(\Delta a^\dagger)^2\rangle}\,. \tag{2.2.8}$$

Inverting these expressions gives the second-order noise moments of a in terms of both Γ and γ:

$$\langle(\Delta a)^2\rangle = -(1-|\Gamma|^2)^{-1}\Gamma = -(\gamma^*+1)(4\gamma_1)^{-1}(\gamma-1)\,, \tag{2.2.9a}$$

$$\langle|\Delta a|^2\rangle = \tfrac{1}{2}(1+|\Gamma|^2)(1-|\Gamma|^2)^{-1} = (1+|\gamma|^2)(4\gamma_1)^{-1}\,. \tag{2.2.9b}$$

Note also that

$$1-|\Gamma|^2 = 4\gamma_1|\gamma+1|^{-2}\,; \tag{2.2.10}$$

hence normalizability dictates that $|\Gamma|<1$.

From the above relations one can see that only two of the three real pieces of information in the second-order noise moments for a single-mode GPS are independent, since

$$\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle = \tfrac{1}{4} + \langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\text{sym}}^2 = \langle\Delta\hat{x}\,\Delta\hat{p}\rangle\langle\Delta\hat{p}\,\Delta\hat{x}\rangle\,, \tag{2.2.11a}$$

$$\langle|\Delta a|^2\rangle^2 = \tfrac{1}{4} + |\langle(\Delta a)^2\rangle|^2 \tag{2.2.11b}$$

[cf. eqs. (2.1.6)]. These relations are made more obvious below [eqs. (2.2.31), (2.2.32)].

The remaining parameter in the wave function (2.2.1) is the phase angle δ_x; in general it can be any real number. The phase angle δ_x is unobservable, but for a state defined as a particular unitary operator acting on the vacuum state it has a well-defined value, provided one assigns a phase angle to the vacuum-state wave function. The reason that the phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ separates naturally from the rest of the overall phase factor in the wave function lies with the definition (2.1.14) of the single-mode displacement operator. That definition, together with the correspondence $\hat{p}\to -\mathrm{i}\partial_x$ [eq. (2.1.2)], implies that if one "displaces" any single-mode pure state $|\Psi\rangle$, by operating on it with the single-mode displacement operator $D(a,\mu)$, the resulting wave function is related to the original wave function $\langle x|\Psi\rangle$ in the following way:

$$\langle x|D(a,\mu)|\Psi\rangle = \exp(-\tfrac{1}{2}\mathrm{i}p_0x_0)\exp(\mathrm{i}xp_0)\langle x-x_0|\Psi\rangle\,. \tag{2.2.12}$$

Formally, therefore, one way to obtain an arbitrary single-mode pure state $|\Psi_\mu\rangle$ with complex amplitude μ is to operate with the displacement operator $D(a,\mu)$ on a state $|\Psi_0\rangle \equiv U_0|0\rangle$ that has the desired noise properties but has zero complex amplitude ($\langle 0|U_0{}^\dagger a U_0|0\rangle \equiv 0$):

$$|\Psi_\mu\rangle \equiv D(a,\mu)U_0|0\rangle\,. \tag{2.2.13}$$

The property (2.1.16) of the displacement operator then ensures that $|\Psi_\mu\rangle$ has complex amplitude μ,

$$\langle\Psi_\mu|a|\Psi_\mu\rangle = \mu\,. \tag{2.2.14}$$

Any normalized single-mode pure state with complex amplitude μ can be defined by an expression like (2.2.13). The advantage of this definition is that the state's mean values x_0 and p_0 (or the complex amplitude μ) are determined solely by the displacement operator $D(a,\mu)$, and its noise moments of a and $a^\dagger$ are determined solely by the unitary operator U_0. Any normalized single-mode GPS $|\mu_g\rangle$ with complex amplitude μ can therefore be formally defined by

$$|\mu_g\rangle \equiv D(a,\mu)U_g|0\rangle\,. \tag{2.2.15}$$

Note the following three properties of U_g: First, it is uniquely defined only up to (right-hand) multiplication by a rotation operator $R(\theta)$ and an overall phase factor. Second, since it defines the noise moments of a and $a^\dagger$ (or $\hat{x}$ and $\hat{p}$) for the GPS $|\mu_g\rangle$, it has associated with it two independent real parameters (over and above that of a rotation operator and phase factor). Third, since the state $|\mu_g\rangle$ has complex amplitude μ, the expectation value $\langle 0|U_g{}^\dagger a U_g|0\rangle$ must vanish.

The phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ in the wave function $\langle x|\mu_g\rangle$ is given, from eqs. (2.2.1) and (2.2.12), by

$$\exp(\tfrac{1}{2}\mathrm{i}\delta_x) = \langle x=0|U_g|0\rangle/|\langle x=0|U_g|0\rangle|\,. \tag{2.2.16}$$

The phase angle δ_x has no dependence on the complex amplitude μ, provided U_g does not; any dependence of U_g on μ is artificial, however, in the sense that it does not affect the state's complex amplitude $\langle a\rangle$. Consider, for illustration, the coherent state $|\mu\rangle_{\rm coh} \equiv D(a, \mu)|0\rangle$ [eq. (1.14)], for which the operator U_g is the identity operator. Equation (2.2.12) says that the wave function for the coherent state $|\mu\rangle_{\rm coh}$ is related to the vacuum-state wave function $\langle x|0\rangle$ by

$$\langle x|\mu\rangle_{\rm coh} = \exp(-\tfrac{1}{2}\mathrm{i}p_0x_0)\exp(\mathrm{i}xp_0)\langle x-x_0|0\rangle\,, \tag{2.2.17}$$

so the phase angle δ_x for a coherent-state wave function is just equal to the phase angle δ_0 assigned to the vacuum-state coordinate-space wave function; conventionally, δ_0 is set equal to zero.

2.2.2. *Operators of which single-mode GPS are eigenstates*

The form of its wave function shows that a single-mode GPS $|\mu_g\rangle$ is an eigenstate of an operator g that is proportional to the linear combinations $\hat{x} + \mathrm{i}\gamma^{-1}\hat{p}$ or $a + \Gamma a^\dagger$. The label μ_g for the GPS $|\mu_g\rangle$ is chosen to be the eigenvalue of g. Thus, one can write the following relations:

$$g|\mu_g\rangle = \mu_g|\mu_g\rangle\,, \tag{2.2.18a}$$

$$g \propto \hat{x} + \mathrm{i}\gamma^{-1}\hat{p} \propto a + \Gamma a^\dagger\,, \tag{2.2.18b}$$

$$\mu_g \propto x_0 + \mathrm{i}\gamma^{-1}p_0 \propto \mu + \Gamma\mu^*\,. \tag{2.2.18c}$$

It is instructive to consider the general form for the operators g of which the single-mode GPS $|\mu_g\rangle$ is an eigenstate:

$$\begin{aligned} g &= \rho_c a + \rho_s a^\dagger \equiv \rho_c(a + \Gamma a^\dagger) \\ &= \rho_p\hat{x} + \mathrm{i}\rho_x\hat{p} \equiv \rho_p(\hat{x} + \mathrm{i}\gamma^{-1}\hat{p})\,. \end{aligned} \tag{2.2.19a}$$

Here ρ_p, ρ_x, ρ_c, and ρ_s are complex numbers, related to each other by

$$\rho_{\substack{c\\s}} = 2^{-1/2}(\rho_p \pm \rho_x)\,, \qquad \rho_{\substack{p\\x}} = 2^{-1/2}(\rho_c \pm \rho_s)\,. \tag{2.2.19b}$$

The eigenvalue μ_g is related to the complex amplitude μ and the mean position and momentum by similar relations,

$$\mu_g = \rho_c\mu + \rho_s\mu^* = \rho_p x_0 + \mathrm{i}\rho_x p_0\,. \tag{2.2.20}$$

Inverting eqs. (2.2.19a,b) leads to the following expressions for a and the complex amplitude μ in terms of g and the eigenvalue μ_g:

$$a = [g, g^\dagger]^{-1}(\rho_c^* g - \rho_s g^\dagger)\,, \tag{2.2.21a}$$

$$\mu \equiv \langle a\rangle = [g, g^\dagger]^{-1}(\rho_c^*\mu_g - \rho_s\mu_g^*)\,. \tag{2.2.21b}$$

These relations imply the important equality

$$D(a,\mu) = D(g,[g,g^{\dagger}]^{-1}\mu_g)\,. \tag{2.2.22}$$

The equality (2.2.22) enables one to see explicitly how the form of the unitary operator U_g, which defines a single-mode GPS $|\mu_g\rangle$ through eq. (2.2.15), is determined by the form of the operators g. To see this, begin with an alternative definition for the GPS $|\mu_g\rangle$. First, assume that $|\mu_g\rangle$ is related to the vacuum state by some unitary operator $\bar{U}$:

$$|\mu_g\rangle = \bar{U}|0\rangle\,. \tag{2.2.23a}$$

It is then convenient to define another unitary operator U_g by

$$\bar{U} \equiv U_g D(a,\mu_g)\,, \tag{2.2.23b}$$

so that the state $|\mu_g\rangle$ is equal to the operator U_g acting on the coherent state $|\mu_g\rangle_{\mathrm{coh}}$,

$$|\mu_g\rangle = U_g D(a,\mu_g)|0\rangle = U_g|\mu_g\rangle_{\mathrm{coh}}\,. \tag{2.2.24}$$

[The equality (2.2.22) will be seen to ensure that the operator U_g defined here is the same as that in eq. (2.2.15).] It is then consistent with the eigenvalue equation (2.2.18a) that the operator g be unitarily related to the annihilation operator a through the operator U_g:

$$g = U_g a U_g^{\dagger}\,. \tag{2.2.25}$$

The form of U_g is thus determined by the form of the operators g. The unitarity of U_g ensures that $[g,g^{\dagger}] = [a,a^{\dagger}] = 1$. This, together with the form (2.2.19a) of g, implies the equality

$$D(a,\mu) = D(g,\mu_g) \tag{2.2.26}$$

[eq. (2.2.22)], which in turn proves the equivalence of the definitions (2.2.15) and (2.2.24) for $|\mu_g\rangle$. Thus, any single-mode GPS $|\mu_g\rangle$ has the following two equivalent definitions:

$$|\mu_g\rangle = D(a,\mu)U_g|0\rangle = U_g D(a,\mu_g)|0\rangle \equiv U_g|\mu_g\rangle_{\mathrm{coh}}\,. \tag{2.2.27}$$

An equivalent line of reasoning has been used by Stoler [12] in a discussion of single-mode MUS (single-mode squeezed states with $\varphi = 0$).

Return now to the general forms (2.2.19a) for the operators g of which single-mode GPS are eigenstates. Two of the four real parameters in the expressions (2.2.19a) for g are determined by the wave function $\langle x|\mu_g\rangle$, which specifies the ratios

$$\rho_p/\rho_x \equiv \gamma\,, \qquad \rho_s/\rho_c \equiv \Gamma\,. \tag{2.2.28}$$

[It will be seen that these parameters specify the squeeze factor r and squeeze angle φ.] The third real parameter in g is partially determined by the requirement that g have a complete (or overcomplete) set of normalizable eigenstates, i.e., that the commutator $[g, g^\dagger]$ be a positive real number (see appendix C). It is determined completely if one specifies that g be unitarily related to the annihilation operator a [eq. (2.2.25)], which implies that

$$[g, g^\dagger] = [a, a^\dagger] = 1\,. \tag{2.2.29}$$

The commutator $[g, g^\dagger]$ can be written in the following different ways, using eqs. (2.2.5) and (2.2.8):

$$\begin{aligned}[g, g^\dagger] &= 2\,\mathrm{Re}(\rho_x{}^*\rho_p) = 2|\rho_x|^2\gamma_1 = |\rho_x|^2/\langle(\Delta\hat{x})^2\rangle \\ &= 2|\rho_p|^2\,\mathrm{Re}(\gamma^{-1}) = |\rho_p|^2/\langle(\Delta\hat{p})^2\rangle \\ &= |\rho_c|^2 - |\rho_s|^2 = |\rho_c|^2(1-|\Gamma|^2) = |\rho_c|^2(\langle|\Delta a|^2\rangle + \tfrac{1}{2})^{-1} \\ &= |\rho_s|^2(|\Gamma|^{-2} - 1) = |\rho_s|^2(\langle|\Delta a|^2\rangle - \tfrac{1}{2})^{-1} \\ &= \rho_c^*\rho_s\Gamma^{-1}(1-|\Gamma|^2) = -\rho_c^*\rho_s\langle(\Delta a)^2\rangle^{-1}\end{aligned} \tag{2.2.30}$$

[eqs. (2.2.5) and (2.2.8)]. Thus, the condition that g be unitarily related to a [eq. (2.2.29)] implies that

$$\mathrm{Re}(\rho_x{}^*\rho_p) = \tfrac{1}{2}\,, \qquad |\rho_c|^2 - |\rho_s|^2 = 1\,. \tag{2.2.31}$$

The expressions (2.2.30) show that the operators g have normalizable eigenstates – i.e., $[g, g^\dagger]$ is a positive real number – if and only if $\gamma_1 > 0$, or $|\Gamma| < 1$; this is equivalent to the condition that the wave function $\langle x|\mu_g\rangle$ be normalizable [eq. (2.2.3)]. This condition also requires that the numbers ρ_p, ρ_x, and ρ_c be nonzero. The only remaining parameter in g is an overall multiplicative phase factor. Multiplying g by a phase factor $e^{i\theta}$ is equivalent to multiplying U_g (on the right) by a rotation operator $R(\theta)$. The definition (2.2.27) of $|\mu_g\rangle$ shows that this freedom reflects the fact that a coherent state remains a coherent state when multiplied by a rotation operator [eq. (2.1.20)].

The expressions (2.2.30) for the commutator $[g, g^\dagger]$ reveal the following simple relations between the second-order noise moments of a, $a^\dagger$, $\hat{x}$, and $\hat{p}$ and the numbers ρ_p, ρ_x, ρ_c, and ρ_s that define operators g unitarily related to a:

$$\langle(\Delta\hat{x})^2\rangle = |\rho_x|^2\,, \qquad \langle(\Delta\hat{p})^2\rangle = |\rho_p|^2\,, \qquad \langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}} = -\mathrm{Im}(\rho_x{}^*\rho_p)\,, \tag{2.2.32a}$$

$$\langle(\Delta a)^2\rangle = -\rho_c{}^*\rho_s\,, \tag{2.2.32b}$$

$$\langle|\Delta a|^2\rangle = \tfrac{1}{2}(|\rho_c|^2 + |\rho_s|^2) = |\rho_c|^2 - \tfrac{1}{2} = |\rho_s|^2 + \tfrac{1}{2} \tag{2.2.32c}$$

[cf. eqs. (2.2.6) and (2.2.9)]. These expressions make obvious the equalities (2.2.11) satisfied by the second-order noise moments.

2.2.3. *Unitary relation of single-mode GPS to the vacuum state*

The form of the unitary operator U_g in the definition (2.2.27) of the single-mode GPS $|\mu_g\rangle$ is determined by the transformation (2.2.25) and the form of g [eqs. (2.2.19a), (2.2.31)]. The linearity and absence of any additive constants in the transformation imply that $U_g = \exp[-iH_g^{(1)}t]$, where $H_g^{(1)}$ is a (Hermitian) linear combination of the three operators $a^\dagger a$, a^2, and $a^{\dagger 2}$. That is, the generator $H_g^{(1)}$ of U_g has the general form $H_0^{(1)} + H_2^{(1)}$ defined in the Introduction [eqs. (1.1)–(1.4)]. It is shown in subsection 2.3 and appendix A that the operator U_g can always be written as a product of a single-mode squeeze operator and a single-mode rotation operator (and an unobservable overall phase factor). The rotation operator can be placed either to the right or left of the squeeze operator, without changing the general form of U_g [eq. (2.1.25)]. When placed to the right of the squeeze operator, it acts like the identity operator on the vacuum state and hence is inconsequential. The rotation operator can therefore be neglected in the general form for U_g. Hence the operator U_g that relates the most general single-mode GPS to a single-mode coherent state is a single-mode squeeze operator $S_1(r, \varphi)$. The most general single-mode GPS is that state defined by eq. (2.2.27) with $U_g \equiv S_1(r, \varphi)$; i.e., it is the SMSS

$$|\mu_\alpha\rangle_{(r,\varphi)} \equiv D(a, \mu)S_1(r, \varphi)|0\rangle = S_1(r, \varphi)|\mu_\alpha\rangle_{\mathrm{coh}} \tag{2.2.33}$$

[eq. (1.16)]. A single-mode GPS is thus completely defined by its complex amplitude μ and the values of its two real parameters r and φ.

The SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ is an eigenstate of the squeezed annihilation operator

$$\alpha(r, \varphi) \equiv S_1(r, \varphi)aS_1^\dagger(r, \varphi) = a\cosh r + a^\dagger e^{2i\varphi}\sinh r \tag{2.2.34a}$$

$$= 2^{-1/2}(\cosh r + e^{2i\varphi}\sinh r)\hat{x} + i2^{-1/2}(\cosh r - e^{2i\varphi}\sinh r)\hat{p} \tag{2.2.34b}$$

[eq. (2.1.23a); cf. eq. (2.2.19a)]. The complex numbers ρ_c, ρ_s, Γ and ρ_p, ρ_x, γ, which define the noise moments of a, $a^\dagger$, $\hat{x}$, and $\hat{p}$, are therefore related to r and φ by

$$\rho_c \equiv \cosh r, \qquad \rho_s \equiv e^{2i\varphi}\sinh r; \qquad \rho_{\substack{p\\x}} \equiv 2^{-1/2}(\cosh r \pm e^{2i\varphi}\sinh r); \tag{2.2.35a}$$

$$\Gamma \equiv \rho_s/\rho_c = e^{2i\varphi}\tanh r, \qquad \gamma \equiv \rho_p/\rho_x = (\cosh r + e^{2i\varphi}\sinh r)/(\cosh r - e^{2i\varphi}\sinh r). \tag{2.2.35b}$$

The complex amplitude μ and the eigenvalue μ_α are related to each other by

$$\mu \equiv \langle a\rangle = \mu_\alpha\cosh r - \mu_\alpha^* e^{2i\varphi}\sinh r, \tag{2.2.36a}$$

$$\mu_\alpha = \mu\cosh r + \mu^* e^{2i\varphi}\sinh r \tag{2.2.36b}$$

[eqs. (2.2.20) and (2.2.21b)]. The second-order noise moments of a SMSS in terms of r and φ are obtained by inserting the expressions (2.2.35) for ρ_c, ρ_s, ρ_p, and ρ_x into eqs. (2.2.32); they were given in eqs. (2.1.34) and (2.1.35).

The phase angle δ_x in the coordinate-space wave function for the SMSS $|\mu_\alpha\rangle_{(r,\varphi)}$ can be obtained from eq. (2.2.16). The calculation is described in appendix B. The result is

$$\exp(\tfrac{1}{2}i\delta_x) = \frac{(\cosh r - e^{-2i\varphi}\sinh r)^{1/2}}{|\cosh r - e^{-2i\varphi}\sinh r|^{1/2}} \equiv \frac{(\rho_x^*)^{1/2}}{|\rho_x|^{1/2}}. \tag{2.2.37}$$

2.2.4. Single-mode momentum-space Gaussian wave function

To conclude this discussion of single-mode Gaussian wave functions, consider briefly the momentum-space wave function for a single-mode Gaussian pure state, $\langle p|\mu_g\rangle$, obtained by Fourier transforming $\langle x|\mu_g\rangle$ [eq. (2.2.1)]; here the dimensionless momentum variable p is the eigenvalue of the Hermitian operator $\hat{p}$. The momentum-space wave function has the following form:

$$\langle p|\mu_g\rangle \equiv (2\pi)^{-1/2}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px}\langle x|\mu_g\rangle$$

$$\equiv \mathcal{N}_g \exp(-\tfrac{1}{2}\mathrm{i}\delta_p)\exp(\tfrac{1}{2}\mathrm{i}p_0x_0)\exp(-\mathrm{i}x_0p)\exp[-(p-p_0)^2/2\gamma]\,, \tag{2.2.38a}$$

where the (real) normalization constant $\mathcal{N}_g$ is

$$\mathcal{N}_g = (\pi|\gamma|^2/\gamma_1)^{-1/4} = (2\pi\langle(\Delta\hat{p})^2\rangle)^{-1/4} \tag{2.2.38b}$$

[cf. eqs. (2.2.1) and (2.2.7)]. The phase angle δ_p is related to the coordinate-space phase angle δ_x by

$$\exp(\mathrm{i}\delta_p) = \exp(-\mathrm{i}\delta_x)\frac{\gamma}{|\gamma|} = \exp(-\mathrm{i}\delta_x)\frac{-\mathrm{i}(\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}}+\frac{1}{2}\mathrm{i})}{|\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}}+\frac{1}{2}\mathrm{i}|}\,. \tag{2.2.39}$$

For the single-mode squeezed state $|\mu_\alpha\rangle_{(r,\varphi)}$ the phase factor $\exp(-\frac{1}{2}\mathrm{i}\delta_p)$ is therefore

$$\exp(-\tfrac{1}{2}\mathrm{i}\delta_p) = \frac{(\cosh r + \mathrm{e}^{-2\mathrm{i}\varphi}\sinh r)^{1/2}}{|\cosh r + \mathrm{e}^{-2\mathrm{i}\varphi}\sinh r|^{1/2}} = \frac{(\rho_p^*)^{1/2}}{|\rho_p|^{1/2}}\,. \tag{2.2.40}$$

The position and momentum probabilities have the usual Gaussian forms:

$$|\langle x|\mu_g\rangle|^2 = (2\pi\langle(\Delta\hat{x})^2\rangle)^{-1/2}\exp[-(x-x_0)^2/2\langle(\Delta\hat{x})^2\rangle]\,, \tag{2.2.41a}$$

$$|\langle p|\mu_g\rangle|^2 = (2\pi\langle(\Delta\hat{p})^2\rangle)^{-1/2}\exp[-(p-p_0)^2/2\langle(\Delta\hat{p})^2\rangle]\,. \tag{2.2.41b}$$

2.3. Two-component vector notation for single-mode GPS

This section describes a two-component vector notation that serves as the basis for an efficient and powerful way of characterizing all single-mode states (pure or mixed) with Gaussian noise statistics. Subsection 2.3.1 defines the fundamental vectors and two-dimensional matrices. Subsection 2.3.2 writes the unitary operators and transformations associated with single-mode GPS in the vector notation, and uses the vector notation to derive many useful properties of the single-mode displacement, rotation, and squeeze operators. Subsection 2.3.3 discusses the group theoretical significances of the transformation matrices that arise from unitary transformations by single-mode rotation and squeeze operators. Subsection 2.3.4 defines two-dimensional second-order noise matrices and discusses some of their important properties. Subsection 2.3.5 uses the vector notation to derive the unitary evolution operator associated with the most general combination of interaction Hamiltonians that can produce a single-mode GPS. This operator is shown to be expressible as the product of single-mode displacement, squeeze, and rotation operators. It is thus proved rigorously that the most general single-mode GPS is a single-mode squeezed state.

2.3.1. Fundamental vectors and matrices

The previous discussion has shown that the unitary operators that relate single-mode GPS to the vacuum state and to other single-mode GPS are rotation, displacement, and single-mode squeeze operators. Since these operators induce linear transformations on a and $a^\dagger$ (or $\hat{x}$ and $\hat{p}$), it is useful to define the two-component operator column vectors [29, 16, 33, 20]

$$\boldsymbol{a}_s \equiv \begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \qquad \hat{\boldsymbol{x}}_s \equiv \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = A\boldsymbol{a}_s, \tag{2.3.1a}$$

$$A \equiv 2^{-1/2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} = (A^\dagger)^{-1}. \tag{2.3.1b}$$

The expectation values of these operator column vectors are column vectors whose components are complex numbers (for $\boldsymbol{a}_s$), or real numbers (for $\hat{\boldsymbol{x}}_s$):

$$\boldsymbol{\mu}_s \equiv \langle \boldsymbol{a}_s \rangle = \begin{pmatrix} \mu \\ \mu^* \end{pmatrix}, \qquad \boldsymbol{\xi}_s \equiv \langle \hat{\boldsymbol{x}}_s \rangle = \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} = A\boldsymbol{\mu}_s. \tag{2.3.2}$$

The adjoints of the operator column vectors are the row vectors

$$\boldsymbol{a}_s^\dagger \equiv (a^\dagger a), \qquad \hat{\boldsymbol{x}}_s^\dagger \equiv (\hat{x}\,\hat{p}) = \hat{\boldsymbol{x}}_s^{\mathrm{T}}, \tag{2.3.3}$$

where a superscript "T" means transpose. The transpose of the adjoint of an operator column vector is denoted by a superscript "*":

$$(\boldsymbol{a}_s^\dagger)^{\mathrm{T}} \equiv \boldsymbol{a}_s^* = \begin{pmatrix} a^\dagger \\ a \end{pmatrix}, \qquad (\hat{\boldsymbol{x}}_s^\dagger)^{\mathrm{T}} \equiv \hat{\boldsymbol{x}}_s^* = \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = \hat{\boldsymbol{x}}_s. \tag{2.3.4}$$

Similar definitions hold for column vectors of complex numbers. Note that the product of a column vector and a row vector, e.g., $\boldsymbol{a}_s\boldsymbol{a}_s^\dagger$, is a tensor product (i.e., a two-dimensional matrix), whereas the product of a row vector and a column vector, e.g., $\boldsymbol{a}_s^\dagger\boldsymbol{a}_s$, is a scalar product (i.e., an operator or number).

There are three two-dimensional matrices, in addition to the identity matrix $\mathbf{1}$, that arise naturally with this vector notation. They are the Pauli matrices σ_1, σ_2, σ_3,

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 \equiv \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3.5a}$$

They satisfy

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + \mathrm{i}\varepsilon_{ijk}\sigma_k, \qquad i, j, k = 1, 2, 3. \tag{2.3.5b}$$

It is useful to define rotated versions of σ_1 and σ_2:

$$\sigma_\varphi \equiv \sigma_1 \cos 2\varphi - \sigma_2 \sin 2\varphi = \begin{pmatrix} 0 & e^{2i\varphi} \\ e^{-2i\varphi} & 0 \end{pmatrix}, \tag{2.3.6a}$$

$$\sigma_{\varphi-\pi/4} = \sigma_1 \sin 2\varphi + \sigma_2 \cos 2\varphi = \begin{pmatrix} 0 & -ie^{2i\varphi} \\ ie^{-2i\varphi} & 0 \end{pmatrix}. \tag{2.3.6b}$$

Note that $[\sigma_\varphi, \sigma_{\varphi-\pi/4}] = [\sigma_1, \sigma_2] = 2i\sigma_3$.

The commutation relations for a, $a^\dagger$ and $\hat{x}$, $\hat{p}$ are conveniently expressed by the Hermitian commutator matrices

$$[\boldsymbol{a}_s, \boldsymbol{a}_s^\dagger] \equiv \boldsymbol{a}_s\boldsymbol{a}_s^\dagger - (\boldsymbol{a}_s^*\boldsymbol{a}_s^{\mathrm{T}})^{\mathrm{T}} = \sigma_3\,, \tag{2.3.7a}$$

$$[\hat{\boldsymbol{x}}_s, \hat{\boldsymbol{x}}_s^{\mathrm{T}}] \equiv \hat{\boldsymbol{x}}_s\hat{\boldsymbol{x}}_s^{\mathrm{T}} - (\hat{\boldsymbol{x}}_s\hat{\boldsymbol{x}}_s^{\mathrm{T}})^{\mathrm{T}} = A\sigma_3 A^\dagger = -\sigma_2\,. \tag{2.3.7b}$$

2.3.2. Unitary operators and transformations

The (single-mode) rotation, displacement, and squeeze operators are expressed in vector notation by

$$R(\theta) \equiv \exp[-i\theta a^\dagger a] = \exp(\tfrac{1}{2}i\theta)\exp[-\tfrac{1}{2}i\theta \boldsymbol{a}_s^\dagger \boldsymbol{a}_s]\,, \tag{2.3.8a}$$

$$D(a, \mu) \equiv \exp[\mu a^\dagger - \mu^* a] = \exp[\boldsymbol{a}_s^\dagger \sigma_3 \boldsymbol{\mu}_s]\,, \tag{2.3.8b}$$

$$S_1(r, \varphi) \equiv \exp[\tfrac{1}{2}r(e^{-2i\varphi}a^2 - e^{2i\varphi}a^{\dagger 2})] = \exp[-\tfrac{1}{2}ir\boldsymbol{a}_s^\dagger \sigma_{\varphi-\pi/4}\boldsymbol{a}_s]\,. \tag{2.3.8c}$$

A unitary transformation by the displacement operator on the components of the column vectors $\boldsymbol{a}_s$ or $\hat{\boldsymbol{x}}_s$ results in the addition of a constant column vector:

$$D(a, \mu)\boldsymbol{a}_s D^\dagger(a, \mu) = \boldsymbol{a}_s - \boldsymbol{\mu}_s\,, \qquad D(a, \mu)\hat{\boldsymbol{x}}_s D^\dagger(a, \mu) = \hat{\boldsymbol{x}}_s - \boldsymbol{\xi}_s \tag{2.3.9}$$

[eq. (2.1.16)]. Unitary transformations by rotation operators and squeeze operators result in matrix transformations of $\boldsymbol{a}_s$ and $\hat{\boldsymbol{x}}_s$. An easy way to obtain these transformation matrices is to note the following general relation, for arbitrary two-dimensional matrix K, which follows from the commutator matrix $[\boldsymbol{a}_s, \boldsymbol{a}_s^\dagger] = \sigma_3$:

$$[\boldsymbol{a}_s^\dagger K \boldsymbol{a}_s, \boldsymbol{a}_s] = K_0 \boldsymbol{a}_s\,, \qquad K_0 \equiv -\sigma_3(K + \sigma_1 K^{\mathrm{T}}\sigma_1) = -\sigma_3 \operatorname{Tr} K + [K, \sigma_3]\,. \tag{2.3.10a}$$

This implies that

$$\exp(\boldsymbol{a}_s^\dagger K \boldsymbol{a}_s)\boldsymbol{a}_s \exp(-\boldsymbol{a}_s^\dagger K \boldsymbol{a}_s) = e^{K_0}\boldsymbol{a}_s\,. \tag{2.3.10b}$$

The matrix transformations induced on the column vectors $\boldsymbol{a}_s$ and $\hat{\boldsymbol{x}}_s$ by the rotation operator $R(\theta)$ are therefore

$$R(\theta)\boldsymbol{a}_s R^\dagger(\theta) = \begin{pmatrix} a(\theta) \\ a^\dagger(\theta) \end{pmatrix} = \exp(\mathrm{i}\theta\sigma_3)\boldsymbol{a}_s \equiv \boldsymbol{a}_s(\theta)\,, \tag{2.3.11a}$$

$$R(\theta)\hat{\boldsymbol{x}}_s R^\dagger(\theta) = \begin{pmatrix} \hat{x}(\theta) \\ \hat{p}(\theta) \end{pmatrix} = A\exp(\mathrm{i}\theta\sigma_3)A^\dagger\hat{\boldsymbol{x}}_s = \exp(-\mathrm{i}\theta\sigma_2)\hat{\boldsymbol{x}}_s \equiv \hat{\boldsymbol{x}}_s(\theta)\,; \tag{2.3.11b}$$

$$\exp(\mathrm{i}\theta\sigma_3) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}, \qquad \exp(-\mathrm{i}\theta\sigma_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.3.11c}$$

[eqs. (2.1.10)].

The matrix transformations induced on the column vectors $\boldsymbol{a}_s$ and $\hat{\boldsymbol{x}}_s$ by the single-mode squeeze operator $S_1(r, \varphi)$ are

$$S_1(r,\varphi)\boldsymbol{a}_s S_1^\dagger(r,\varphi) = \begin{pmatrix} \alpha(r,\varphi) \\ \alpha^\dagger(r,\varphi) \end{pmatrix} = C_{r,\varphi}\boldsymbol{a}_s \equiv \boldsymbol{\alpha}_s(r,\varphi)\,, \tag{2.3.12a}$$

$$C_{r,\varphi} \equiv \begin{pmatrix} \cosh r & \mathrm{e}^{2\mathrm{i}\varphi}\sinh r \\ \mathrm{e}^{-2\mathrm{i}\varphi}\sinh r & \cosh r \end{pmatrix} = \cosh r\,\mathbf{1} + \sinh r\,\sigma_\varphi = \exp(r\sigma_\varphi)\,; \tag{2.3.12b}$$

$$S_1(r,\varphi)\hat{\boldsymbol{x}}_s S_1^\dagger(r,\varphi) = AC_{r,\varphi}A^\dagger\hat{\boldsymbol{x}}_s\,, \tag{2.3.13a}$$

$$\begin{aligned} AC_{r,\varphi}A^\dagger &\equiv \begin{pmatrix} \cosh r + \sinh r\cos 2\varphi & \sinh r\sin 2\varphi \\ \sinh r\sin 2\varphi & \cosh r - \sinh r\cos 2\varphi \end{pmatrix} \\ &= \exp[r(\sigma_3\cos 2\varphi + \sigma_1\sin 2\varphi)] \end{aligned} \tag{2.3.13b}$$

[eq. (2.3.1b)]. The Hermitian matrix $C_{r,\varphi}$ has the following important properties:

$$C_{r,\varphi}^{-1} = C_{-r,\varphi} = C_{r,\varphi+\pi/2} = \sigma_3 C_{r,\varphi}\sigma_3\,; \tag{2.3.14a}$$

$$C_{r,0} = \exp(r\sigma_1) = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}; \qquad C_{r,\pi/4} = \exp(-r\sigma_2) = \begin{pmatrix} \cosh r & \mathrm{i}\sinh r \\ -\mathrm{i}\sinh r & \cosh r \end{pmatrix}; \tag{2.3.14b}$$

$$C_{r,\varphi} = \exp(\mathrm{i}\varphi'\sigma_3)C_{r,\varphi-\varphi'}\exp(-\mathrm{i}\varphi'\sigma_3)\,; \tag{2.3.14c}$$

$$C_{r,\varphi}C_{r',\varphi} = C_{r+r',\varphi} \tag{2.3.14d}$$

(see also appendix A of ref. [20]).

Many properties of the single-mode squeeze operator $S_1(r, \varphi)$ that would otherwise be difficult to see can be found from properties of the transformation matrix $C_{r,\varphi}$. For example, one can factor $S_1(r, \varphi)$ into a product of exponentials of the operators $\frac{1}{2}a^2$, $\frac{1}{2}a^{\dagger 2}$, and $(aa^\dagger)_{\mathrm{sym}}$ simply by factoring the matrix $C_{r,\varphi}$ into exponentials of matrices (linear combinations of the Pauli matrices) that have a commutator algebra identical to that of these operators (see, e.g., refs. [15, 48] or [20]). Examples of such matrices are $-\sigma_-$, σ_+, and σ_3, where $\sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. These factored forms are listed in appendix B [eq. (B.12)] of ref. [20]; one of the most useful is

$$S_1(r, \varphi) = (\cosh r)^{-1/2} \exp(-\tfrac{1}{2}\Gamma a^{\dagger 2}) \exp[-\ln(\cosh r) a^\dagger a] \exp(\tfrac{1}{2}\Gamma^* a^2), \qquad \Gamma \equiv e^{2i\varphi} \tanh r. \tag{2.3.15}$$

It is useful, for example, in expressing a SMSS as a sum over number states (a technique useful for calculating the phase factor $\exp(\frac{1}{2}i\delta_x)$ in the wave function for a SMSS; see appendix B). Also, the product of two different squeeze operators can be found from the product of two different matrices $C_{r,\varphi}$. In this way one finds that any product of single-mode squeeze operators can be expressed as the product of one single-mode squeeze operator and one rotation operator, by use of the following rule:

$$S_1^\dagger(r', \varphi') S_1(r, \varphi) = e^{-i\theta} R(\theta) S_1(\bar{r}, \bar{\varphi}) = e^{-i\theta} S_1(\bar{r}, \bar{\varphi} - \theta) R(\theta). \tag{2.3.16a}$$

The real numbers θ, $\bar{r}$, and $\bar{\varphi}$ are related to r', φ', r, and φ by the matrix equality

$$C_{\bar{r},\bar{\varphi}} \exp(i\theta\sigma_3) = \begin{pmatrix} e^{i\theta} \cosh \bar{r} & e^{i(2\bar{\varphi}-\theta)} \sinh \bar{r} \\ e^{-i(2\bar{\varphi}-\theta)} \sinh \bar{r} & e^{-i\theta} \cosh \bar{r} \end{pmatrix} = C_{r,\varphi} C_{-r',\varphi'} \tag{2.3.16b}$$

[eq. (B14) of ref. [20]]. For the special case $\varphi = \varphi'$ one finds the simple relation

$$S_1(r, \varphi) S_1(r', \varphi) = S_1(r + r', \varphi) \tag{2.3.16c}$$

[cf. eq. (2.3.14d)].

The vector notation simplifies the task of ordering noncommuting unitary operators. For example, using the matrix transformations given above one finds that

$$R(\theta) D(a, \mu) = \exp[\boldsymbol{a}_s^\dagger \exp(-i\theta\sigma_3)\sigma_3 \boldsymbol{\mu}_s] R(\theta) = D(a, e^{-i\theta}\mu) R(\theta), \tag{2.3.17a}$$

$$\begin{aligned} R(\theta) S_1(r, \varphi) &= \exp[\tfrac{1}{2} r \boldsymbol{a}_s^\dagger \exp(-i\theta\sigma_3) \sigma_{\varphi-\pi/4} \exp(i\theta\sigma_3) \boldsymbol{a}_s] R(\theta) \\ &= \exp[\tfrac{1}{2} r \boldsymbol{a}_s^\dagger \sigma_{\varphi-\theta-\pi/4} \boldsymbol{a}_s] R(\theta) = S_1(r, \varphi - \theta) R(\theta), \end{aligned} \tag{2.3.17b}$$

$$\begin{aligned} D(a, \mu) S_1(r, \varphi) &= S_1(r, \varphi) \exp[\boldsymbol{a}_s^\dagger C_{-r,\varphi} \sigma_3 \boldsymbol{\mu}_s] \\ &= S_1(r, \varphi) \exp[\boldsymbol{a}_s^\dagger \sigma_3 C_{r,\varphi} \boldsymbol{\mu}_s] \equiv S_1(r, \varphi) D(a, \mu_\alpha), \\ \boldsymbol{\mu}_{\alpha s} \equiv \begin{pmatrix} \mu_\alpha \\ \mu_\alpha^* \end{pmatrix} &\equiv C_{r,\varphi} \boldsymbol{\mu}_s \end{aligned} \tag{2.3.17c}$$

[cf. eqs. (2.1.17), (2.1.25), (2.1.26)].

2.3.3. *Group theoretical properties of transformation matrices*

The transformation matrices (2.3.11)–(2.3.13) arise naturally, without specific reference to the single-mode rotation and squeeze operators, from the requirement that a unitary transformation on a (or $\hat{x}$ and $\hat{p}$) preserve the commutators (2.3.7). Consider, for example, the real, two-dimensional matrices $\bar{M}$ that describe transformations induced on the components of the real column vector $\hat{\boldsymbol{x}}_s$ by a unitary operator U: $\bar{M}\hat{\boldsymbol{x}}_s \equiv U\hat{\boldsymbol{x}}_s U^\dagger$ (this approach has also been used by Milburn, ref. [33]). The unitarity of U implies that the matrices $\bar{M}$ preserve the antisymmetric commutator matrix $[\hat{\boldsymbol{x}}_s, \hat{\boldsymbol{x}}_s^{\mathrm{T}}] = -\sigma_2$ [eq. (2.3.7b)]:

$$\bar{M}\sigma_2\bar{M}^{\mathrm{T}} = \sigma_2 = \bar{M}^{\mathrm{T}}\sigma_2\bar{M}\,. \tag{2.3.18a}$$

The real matrices $\bar{M}$ that satisfy this condition have unity determinant. They comprise the three-parameter symplectic group Sp(2, R) [41]. Transformations induced by the same unitary operator U on the components of the column vector $\boldsymbol{a}_s = A^\dagger\hat{\boldsymbol{x}}_s$ are described by complex two-dimensional matrices M, $M\boldsymbol{a}_s \equiv U\boldsymbol{a}_s U^\dagger$. The matrices M are unitarily related to the real matrices $\bar{M}$ through the matrix A [eq. (2.3.1b)]:

$$M = A^\dagger\bar{M}A\,. \tag{2.3.18b}$$

The matrices M have unity determinant and preserve the Hermitian commutator matrix $[\boldsymbol{a}_s, \boldsymbol{a}_s^\dagger] = \sigma_3$ [eq. (2.3.7a)]:

$$M\sigma_3M^\dagger = \sigma_3 = M^\dagger\sigma_3M\,. \tag{2.3.18c}$$

They comprise the three-parameter, noncompact Lie group SU(1, 1) [41], isomorphic to Sp(2, R).

The linearity of these matrix transformations implies that the unitary operator U is an exponential of the three bilinear combinations of a and $a^\dagger$ (or $\hat{x}$ and $\hat{p}$) – a^2, $a^{\dagger 2}$, and $a^\dagger a$ [or $\hat{x}^2$, $\hat{p}^2$, and $(\hat{x}\hat{p})_{\mathrm{sym}}$]. The three real parameters that characterize the transformation matrices M and $\bar{M}$ are thus related to the three parameters of the unitary operator, i.e., to the coefficients of the three bilinear combinations. The most general such unitary operator can be expressed as a product of a single-mode squeeze and rotation operator, $U = S_1(r, \varphi)R(\theta)$. Hence, from the preceding discussion of the single-mode squeeze and rotation operators, the transformation matrices M have the general form

$$M = \exp(\mathrm{i}\theta\sigma_3)C_{r,\varphi} = C_{r,\varphi+\theta}\exp(-\mathrm{i}\theta\sigma_3) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta}\cosh r & \mathrm{e}^{\mathrm{i}(2\varphi+\theta)}\sinh r \\ \mathrm{e}^{-\mathrm{i}(2\varphi+\theta)}\sinh r & \mathrm{e}^{-\mathrm{i}\theta}\cosh r \end{pmatrix}, \tag{2.3.18d}$$

where θ, r, and φ are real, continuous parameters [eqs. (2.3.11)–(2.3.13)].

It is instructive to obtain the general form (2.3.18d) for the matrices M in another way. Begin by noting that any two-dimensional matrix M that describes a linear transformation on the components of the column vector $\boldsymbol{a}_s$ must satisfy

$$M^* = \sigma_1M\sigma_1\,, \tag{2.3.19a}$$

since $\boldsymbol{a}_s = \sigma_1\boldsymbol{a}_s^*$. This implies that the matrix M has the general form

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \tag{2.3.19b}$$

where α and β are complex numbers. It also implies the following equality:

$$M\sigma_3 M^\dagger \sigma_3 = M\sigma_2 M^{\mathrm{T}}\sigma_2 \,. \tag{2.3.19c}$$

Because the matrix M describes a unitary transformation on the components of $\boldsymbol{a}_s$, it preserves both the Hermitian commutator matrix $[\boldsymbol{a}_s, \boldsymbol{a}_s^{\dagger}] = \sigma_3$ [eq. (2.3.18c)] and the antisymmetric commutator matrix $[\boldsymbol{a}_s, \boldsymbol{a}_s^{\mathrm{T}}] = \mathrm{i}\sigma_2$; i.e., both expressions in eq. (2.3.19c) are equal to the identity matrix. But any two-dimensional matrix satisfies

$$M\sigma_2 M^{\mathrm{T}}\sigma_2 = (\det M)\mathbf{1} \,. \tag{2.3.19d}$$

Hence unitarity implies that $\det M = 1$. The matrices M that describe unitary transformations on the components of the column vector $\boldsymbol{a}_s$ therefore have the general form (2.3.19b) with unity determinant – i.e., they have the general form (2.3.18d).

2.3.4. Second-order noise matrices

The vector notation is particularly useful for calculating second-order noise moments of a, $a^\dagger$, $\hat{x}$, and $\hat{p}$. The two-dimensional matrix that contains all second-order noise moments of a and $a^\dagger$ is the Hermitian matrix

$$\mathcal{Q}_s \equiv \langle \Delta \boldsymbol{a}_s\, \Delta \boldsymbol{a}_s^{\dagger}\rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}(\langle \Delta \boldsymbol{a}_s\, \Delta \boldsymbol{a}_s^{\dagger}\rangle + \langle \Delta \boldsymbol{a}_s^{*}\, \Delta \boldsymbol{a}_s^{\mathrm{T}}\rangle^{\mathrm{T}})$$

$$= \begin{pmatrix} \langle |\Delta a|^2\rangle & \langle (\Delta a)^2\rangle \\ \langle (\Delta a^\dagger)^2\rangle & \langle |\Delta a|^2\rangle \end{pmatrix} = \mathcal{Q}_s^{\dagger} \,. \tag{2.3.20}$$

The two-dimensional matrix that contains all second-order noise moments of $\hat{x}$ and $\hat{p}$ is the real, symmetric covariance matrix

$$\mathcal{S}_s \equiv \langle \Delta \hat{\boldsymbol{x}}_s\, \Delta \hat{\boldsymbol{x}}_s^{\mathrm{T}}\rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}(\langle \Delta \hat{\boldsymbol{x}}_s\, \Delta \hat{\boldsymbol{x}}_s^{\mathrm{T}}\rangle + \langle \Delta \hat{\boldsymbol{x}}_s\, \Delta \hat{\boldsymbol{x}}_s^{\mathrm{T}}\rangle^{\mathrm{T}})$$

$$= \begin{pmatrix} \langle (\Delta \hat{x})^2\rangle & \langle \Delta \hat{x}\, \Delta \hat{p}\rangle_{\mathrm{sym}} \\ \langle \Delta \hat{x}\, \Delta \hat{p}\rangle_{\mathrm{sym}} & \langle (\Delta \hat{p})^2\rangle \end{pmatrix} = A\mathcal{Q}_s A^\dagger = \mathcal{S}_s^{*} = \mathcal{S}_s^{\mathrm{T}} \,. \tag{2.3.21}$$

The relations (2.2.11) imply that for single-mode GPS these matrices satisfy

$$\mathcal{Q}_s \sigma_3 \mathcal{Q}_s \sigma_3 = \tfrac{1}{4}\mathbf{1} \,, \tag{2.3.22a}$$

$$\mathcal{S}_s \sigma_2 \mathcal{S}_s \sigma_2 = \tfrac{1}{4}\mathbf{1} \,. \tag{2.3.22b}$$

Hence their determinants are equal to $\frac{1}{4}$. For a coherent state, both are proportional to the identity matrix:

$$\mathcal{Q}_{s\,\mathrm{coh}} = \mathcal{S}_{s\,\mathrm{coh}} = \tfrac{1}{2}\mathbf{1} \tag{2.3.23}$$

[eqs. (2.1.33)].

The noise matrices $\mathcal{Q}_s$ and $\mathcal{S}_s$ for a state $|\Psi\rangle$ are related to those of the rotated state $R(\theta)|\Psi\rangle$ in the following ways:

$$\langle R^\dagger(\theta)(\Delta \boldsymbol{a}_s\, \Delta \boldsymbol{a}_s^\dagger)_{\rm sym} R(\theta)\rangle = \langle \Delta \boldsymbol{a}_s(-\theta)\, \Delta \boldsymbol{a}_s^\dagger(-\theta)\rangle_{\rm sym} = \exp(-\mathrm{i}\theta\sigma_3)\, \varnothing_s \exp(\mathrm{i}\theta\sigma_3)$$

$$= \begin{pmatrix} \langle|\Delta a|^2\rangle & \mathrm{e}^{-2\mathrm{i}\theta}\langle(\Delta a)^2\rangle \\ \mathrm{e}^{2\mathrm{i}\theta}\langle(\Delta a^\dagger)^2\rangle & \langle|\Delta a|^2\rangle \end{pmatrix} \equiv \varnothing_s(-\theta)\,; \tag{2.3.24a}$$

$$\langle R^\dagger(\theta)(\Delta \hat{\boldsymbol{x}}_s\, \Delta \hat{\boldsymbol{x}}_s^{\rm T})_{\rm sym} R(\theta)\rangle = \langle \Delta \hat{\boldsymbol{x}}_s(-\theta)\, \Delta \hat{\boldsymbol{x}}_s^{\rm T}(-\theta)\rangle_{\rm sym}$$

$$= \exp(\mathrm{i}\theta\sigma_2)\, \mathcal{S}_s \exp(-\mathrm{i}\theta\sigma_2) \equiv \mathcal{S}_s(-\theta)\,. \tag{2.3.24b}$$

They are related to those of the transformed state $S_1(r, \varphi)|\Psi\rangle$ by

$$\langle S_1{}^\dagger(r, \varphi)(\Delta \boldsymbol{a}_s\, \Delta \boldsymbol{a}_s^\dagger)_{\rm sym} S_1(r, \varphi)\rangle = C_{-r,\varphi}\, \varnothing_s C_{-r,\varphi}\,, \tag{2.3.25a}$$

$$\langle S_1{}^\dagger(r, \varphi)(\Delta \hat{\boldsymbol{x}}_s\, \Delta \hat{\boldsymbol{x}}_s^{\rm T})_{\rm sym} S_1(r, \varphi)\rangle = A C_{-r,\varphi} A^\dagger \mathcal{S}_s A C_{-r,\varphi} A^\dagger\,. \tag{2.3.25b}$$

This immediately tells one, for example, that the noise matrices for a single-mode squeezed state $|\mu_\alpha\rangle_{r,\varphi}$ are

$$\varnothing_s = \tfrac{1}{2} C^2_{-r,\varphi} = \tfrac{1}{2} C_{-2r,\varphi} = \tfrac{1}{2}\sigma_3 C_{2r,\varphi}\sigma_3\,, \tag{2.3.26a}$$

$$\mathcal{S}_s = \tfrac{1}{2} A C_{-2r,\varphi} A^\dagger = \tfrac{1}{2}\sigma_2 A C_{2r,\varphi} A^\dagger \sigma_2 \tag{2.3.26b}$$

[eq. (2.3.23); cf. eqs. (2.1.34), (2.1.35)]. The squeezing effect is clearly visible in the transformation of the noise matrix $\mathcal{S}_s$ [eq. (2.3.25b)]. When $\varphi = 0$, this transformation says that

$$\langle S_1{}^\dagger(r, 0)(\Delta \hat{\boldsymbol{x}}_s\, \Delta \hat{\boldsymbol{x}}_s^{\rm T})_{\rm sym} S_1(r, 0)\rangle = \exp(-r\sigma_3)\, \mathcal{S}_s \exp(-r\sigma_3)$$

$$= \begin{pmatrix} \mathrm{e}^{-2r}\langle(\Delta \hat{x})^2\rangle & \langle \Delta \hat{x}\, \Delta \hat{p}\rangle_{\rm sym} \\ \langle \Delta \hat{x}\, \Delta \hat{p}\rangle_{\rm sym} & \mathrm{e}^{2r}\langle(\Delta \hat{p})^2\rangle \end{pmatrix}. \tag{2.3.27}$$

2.3.5. Evolution operator for general single-mode GPS

Finally, the vector notation enables one to see with relative ease how the unitary operator whose generator is a linear combination of the Hermitian forms $H_{\rm R}{}^{(1)}$, $H_1{}^{(1)}$, and $H_2{}^{(1)}$ factors into the product of a single-mode squeeze, rotation and displacement operator (and an overall phase factor). By giving these generators arbitrary time dependences, one can calculate the evolution operator associated with the most general combination of Hamiltonians that can produce single-mode GPS. This result is given here briefly; details that are important for the calculation are presented in appendix A. Equivalent results have been obtained by Yuen [16].

The single-mode rotation Hamiltonian $H_{\rm R}{}^{(1)}(t)$ is expressed in vector notation by

$$H_{\rm R}{}^{(1)}(t) \equiv \omega(t) a^\dagger a = \tfrac{1}{2}\omega(t)(\boldsymbol{a}_s^\dagger \boldsymbol{a}_s - 1)\,, \tag{2.3.28a}$$

where $\omega(t)$ is an arbitrary real-valued function of time t [eqs. (1.3)]. The linear and quadratic

Hamiltonians $H_1^{(1)}(t)$ and $H_2^{(1)}(t)$ are expressed by

$$H_1^{(1)}(t) = \mathrm{i}\boldsymbol{a}_s^\dagger \sigma_3 \boldsymbol{\lambda}_s\,, \qquad \boldsymbol{\lambda}_s \equiv \begin{pmatrix} \lambda(t) \\ \lambda^*(t) \end{pmatrix}; \tag{2.3.28b}$$

$$H_2^{(1)}(t) = \tfrac{1}{2}\kappa(t)\boldsymbol{a}_s^\dagger \sigma_{\varphi_\kappa - \pi/4}\boldsymbol{a}_s\,, \tag{2.3.28c}$$

where λ is a complex-valued function of time, and κ and φ_κ are real-valued functions of time [eqs. (1.4), (2.3.6b)].

The evolution operator $U(t)$ is the solution to the equation

$$\mathrm{i}\partial_t U(t) = [H_\mathrm{R}^{(1)}(t) + H_1^{(1)}(t) + H_2^{(1)}(t)]U(t)\,, \qquad U(0) = 1\,. \tag{2.3.29}$$

It can be written as a product (in any order) of a single-mode squeeze, rotation, and displacement operator, and an overall phase factor. For illustration, consider the following two forms for $U(t)$:

$$U(t) = \mathrm{e}^{\mathrm{i}\delta} D(a, \mu) S_1(r, \varphi) R(\theta) = \mathrm{e}^{\mathrm{i}\delta} S_1(r, \varphi) R(\theta) D(a, \mu_g) \tag{2.3.30}$$

[cf. eq. (2.2.27)]. Here δ, r, φ, and θ are real-valued functions of time, and μ and μ_g are complex-valued functions of time. For notational convenience, explicit reference to the time dependence of these functions is omitted, e.g., $r \equiv r(t)$, etc. The state $U(t)|0\rangle$ is an eigenstate of an operator $g \equiv U(t) a U^\dagger(t)$ (with eigenvalue μ_g), whose relation to a is described by the vector relation

$$\boldsymbol{g}_s \equiv \begin{pmatrix} g \\ g^\dagger \end{pmatrix} = S_1(r, \varphi) R(\theta) \boldsymbol{a}_s R^\dagger(\theta) S_1^\dagger(r, \varphi) = \exp(\mathrm{i}\theta\sigma_3) C_{r,\varphi} \boldsymbol{a}_s \tag{2.3.31a}$$

[eqs. (2.3.11), (2.3.12)]. The complex eigenvalue μ_g is therefore related to the complex amplitude $\mu \equiv \langle a \rangle$ by

$$\boldsymbol{\mu}_{gs} \equiv \begin{pmatrix} \mu_g \\ \mu_g^* \end{pmatrix} \equiv \exp(\mathrm{i}\theta\sigma_3) C_{r,\varphi} \boldsymbol{\mu}_s\,. \tag{2.3.31b}$$

The calculations in appendix A show that the functions r, φ, θ, μ_g (or μ), and δ are related to the Hamiltonian functions κ, φ_κ, ω, and λ by the following matrix, vector, and scalar equalities:

$$\dot{r}\sigma_{\varphi - \pi/4} - \dot{\varphi}\mathbf{1} + (\dot{\varphi} + \dot{\theta}) C_{2r,\varphi} = \omega\mathbf{1} + \kappa\sigma_{\varphi_\kappa - \pi/4}\,, \tag{2.3.32a}$$

$$\dot{\boldsymbol{\mu}}_{gs} = \exp(\mathrm{i}\theta\sigma_3) C_{r,\varphi} \boldsymbol{\lambda}_s \equiv \boldsymbol{\lambda}_{gs}\,, \tag{2.3.32b}$$

$$\dot{\delta} + \tfrac{1}{2}(\dot{\theta} - \omega) = -\tfrac{1}{2}\mathrm{i}\boldsymbol{\mu}_{gs}^\dagger \sigma_3 \dot{\boldsymbol{\mu}}_{gs} = \mathrm{Im}(\mu_g^* \lambda_g) = \mathrm{Im}(\mu^* \lambda)\,. \tag{2.3.32c}$$

(Dots denote derivatives with respect to time.) The initial conditions, dictated by $U(0) = 1$, are

$$\delta(0) = r(0) = \theta(0) = \mu_g(0) = \mu(0) = 0\,. \tag{2.3.33}$$

For illustration, consider the case $\varphi_\kappa \equiv \varphi_{\kappa 0} - \int_0^t \omega(t)\,\mathrm{d}t$, $\varphi_{\kappa 0} \equiv$ constant. Then the matrix equality (2.3.32a) gives

$$r = \int_0^t \kappa(t)\,\mathrm{d}t\,, \qquad \varphi = \varphi_\kappa\,, \qquad \theta = \int_0^t \omega(t)\,\mathrm{d}t\,. \tag{2.3.34}$$

If no driving is present in this case [$\lambda = 0$], then $\mu_g = \delta = 0$, and $U(t) = S_1(r, \varphi_\kappa)R(\theta) = R(\theta)S_1(r, \varphi_{\kappa 0})$. Suppose now that κ is constant, and $\lambda \equiv \lambda_0 \exp[-\mathrm{i}\int_0^t \omega(t)\,\mathrm{d}t]$, $\lambda_0 =$ constant [when $\omega(t) \equiv \Omega =$ constant, this corresponds to driving the oscillator on resonance]. Then $r = \kappa t$, the angles φ and θ are again given by eq. (2.3.34), and the number μ_g is give by the vector relation

$$\boldsymbol{\mu}_{gs} = \int_0^t \mathrm{d}t\, C_{\kappa t,\,\varphi_{\kappa 0}} \boldsymbol{\lambda}_{0s} = \kappa^{-1}[C_{\kappa t,\,\varphi_{\kappa 0}} - 1]\sigma_{\varphi_{\kappa 0}} \boldsymbol{\lambda}_{0s}\,, \qquad \boldsymbol{\lambda}_{0s} \equiv \begin{pmatrix} \lambda_0 \\ \lambda_0^* \end{pmatrix}. \tag{2.3.35}$$

The phase angle δ is

$$\begin{aligned} \delta &= (2\mathrm{i}\kappa)^{-1} \boldsymbol{\lambda}_{0s}^\dagger \sigma_{\varphi_{\kappa 0}} \sigma_3 \left[\int_0^t (1 - C_{\kappa t,\,\varphi_{\kappa 0}})\right] \boldsymbol{\lambda}_{0s} \\ &= \kappa^{-1}(t - \kappa^{-1}\sinh \kappa t)\,\mathrm{Im}[\exp(-2\mathrm{i}\varphi_{\kappa 0})\lambda_0^{*2}]\,. \end{aligned} \tag{2.3.36}$$

3. Two-mode Gaussian pure states

Proceed now to a discussion of two-mode Gaussian pure states – states produced from two harmonic oscillators, each in its ground state, by physical processes whose interaction Hamiltonians are linear or quadratic in the annihilation and creation operators for the two oscillators. The discussion parallels as closely as possible in structure the preceding discussion of single-mode GPS. Subsection 3.1 is an introduction. It concentrates on the definitions and properties of the fundamental unitary operators associated with two-mode GPS, and describes in detail some special kinds of two-mode GPS. Subsection 3.2 considers the most general two-mode Gaussian wave function, and from it shows that the most general two-mode GPS can be expressed as a product of two single-mode squeeze operators and one two-mode squeeze operator acting on a two-mode coherent state. Subsection 3.3 defines and uses a four-component vector notation to develop an efficient and powerful description of all two-mode states (pure or mixed) that have Gaussian noise statistics.

3.1. Introduction

This section is composed of seven subsections. Subsection 3.1.1 is concerned with notation and definitions. It is itself divided into three parts: Part (a) discusses choices for natural "position" and "momentum variables" for two oscillators of different frequencies. Part (b) introduces a two-component vector notation, used throughout this section, that simplifies the description of two-mode GPS and makes it resemble that of single-mode GPS. Part (c) defines two-dimensional matrices of second-order noise moments for two-mode states. Subsections 3.1.2–3.1.6 look closely at the unitary operators that relate two-mode GPS to the (two-mode) vacuum state; these include rotation, mixing, and displacement operators, two single-mode squeeze operators, and a two-mode squeeze operator. Subsection 3.1.7 derives the second-order noise moments of the most general two-mode GPS, and discusses three special kinds of two-mode GPS: two-mode minimum-uncertainty states (MUS), two-mode states with time-stationary (TS) noise, and two-mode states with "time-stationary quadrature-phase" (TSQP) noise.

3.1.1. Notation and definitions

(a) *Dimensionless position and momentum variables*

Consider now two oscillators, with characteristic frequencies ω_+ and ω_- ($\omega_+ \geq \omega_-$). Each oscillator can be described by its own set of annihilation and creation operators – a_+, $a_+^\dagger$ and a_-, $a_-^\dagger$ – or, equivalently, by the dimensionless coordinate and momentum operators $\hat{x}_+$, $\hat{p}_+$ and $\hat{x}_-$, $\hat{p}_-$. These operators are related to each other by

$$\hat{x}_\pm \equiv 2^{-1/2}(a_\pm + a_\pm^\dagger)\,, \qquad \hat{p}_\pm \equiv 2^{-1/2}(-\mathrm{i}a_\pm + \mathrm{i}a_\pm^\dagger)\,, \tag{3.1.1a}$$

$$a_\pm \equiv 2^{-1/2}(\hat{x}_\pm + \mathrm{i}\hat{p}_\pm) \tag{3.1.1b}$$

[eqs. (2.1.1)]. They obey the commutation relations

$$[a_\pm, a_\pm^\dagger] = 1\,, \qquad [\hat{x}_\pm, \hat{p}_\pm] = \mathrm{i}\,; \tag{3.1.2a}$$

$$[a_+, a_-] = [a_+, a_-^\dagger] = [\hat{x}_\pm, \hat{p}_\mp] = [\hat{x}_+, \hat{x}_-] = [\hat{p}_+, \hat{p}_-] = 0\,. \tag{3.1.2b}$$

While the operators $\hat{x}_+$ and $\hat{x}_-$ ($\hat{p}_+$ and $\hat{p}_-$) are both dimensionless, they do not have the same "units", since the natural units of length (momentum) for the two oscillators differ. Dimensionless position and momentum operators that have compatible units for the two oscillators can be obtained by dividing the usual dimensional position and momentum operators by new units of length L_0 and momentum P_0. In general, L_0 and P_0 can be chosen quite arbitrarily, subject to the dimensional restriction $L_0 P_0 = 1$ ($\equiv \hbar$), but there is a natural choice for them. To see this, let the two oscillators be modeled as masses $m_\pm$ on springs (the result will also hold, however, for the normal modes of a quantized field). Their natural units of length and momentum are $(m_\pm\omega_\pm)^{-1/2}$ and $(m_\pm\omega_\pm)^{1/2}$, respectively. Now choose quantities m and Ω, with dimensions of mass and (time)$^{-1}$, respectively, and define $L_0 \equiv (m\Omega)^{-1/2}$, $P_0 \equiv (m\Omega)^{1/2}$. The dimensionless position and momentum operators $Q_\pm$ and $P_\pm$ for this choice are

$$Q_\pm \equiv \lambda_\pm^{-1}\hat{x}_\pm\,, \qquad P_\pm \equiv \lambda_\pm \hat{p}_\pm\,, \tag{3.1.3a}$$

$$\lambda_\pm \equiv P_0^{-1}(m_\pm\omega_\pm)^{1/2} = L_0(m_\pm\omega_\pm)^{1/2} = (m_\pm/m)^{1/2}(\omega_\pm/\Omega)^{1/2}\,. \tag{3.1.3b}$$

Thus, for equal masses with $m_+ = m_- \equiv m$, or for normal modes of a quantized field, the natural choice for $\lambda_\pm$ is $(\omega_\pm/\Omega)^{1/2}$. For modes of the electromagnetic field, these conjugate variables $Q_\pm$ and $P_\pm$ are the

Fourier components of the vector potential and its conjugate momentum, the electric field. The free Hamiltonian for the two modes, written in terms of $Q_\pm$ and $P_\pm$, has the canonical form

$$H_0^{(2)} = \tfrac{1}{2}\Omega[P_+{}^2 + (\omega_+/\Omega)^2 Q_+{}^2 + P_-{}^2 + (\omega_-/\Omega)^2 Q_-{}^2]\,. \tag{3.1.4}$$

In most of this discussion of two-mode Gaussian pure states I use the variables $\hat{x}_\pm$ and $\hat{p}_\pm$, rather than $Q_\pm$ and $P_\pm$, because they provide an easy comparison with the previous discussion of single-mode states. The canonical field variables $Q_\pm$ and $P_\pm$ or, equivalently, the quadrature-phase amplitudes α_1 and α_2 [eqs. (1.21) and (1.22)], are useful for describing the special noise properties of two-mode squeezed states. The variables $Q_\pm$ and $P_\pm$ and the quadrature-phase amplitudes are related by

$$\mathrm{Re}(\alpha_1) = (2\Omega)^{-1}[(\Omega+\varepsilon)Q_+ + (\Omega-\varepsilon)Q_-]\,,$$

$$\mathrm{Re}(\alpha_2) = (2\Omega)^{-1}(P_+ + P_-)\,,$$

$$\mathrm{Im}(\alpha_1) = (2\Omega)^{-1}(P_+ - P_-)\,,$$

$$\mathrm{Im}(\alpha_2) = -(2\Omega)^{-1}[(\Omega+\varepsilon)Q_+ - (\Omega-\varepsilon)Q_-]\,, \tag{3.1.5a}$$

$$\mathrm{Re}(\alpha_j) \equiv \tfrac{1}{2}(\alpha_j + \alpha_j^\dagger)\,, \qquad \mathrm{Im}(\alpha) \equiv -\mathrm{i}\tfrac{1}{2}(\alpha_j - \alpha_j^\dagger)\,, \qquad j = 1,2\,. \tag{3.1.5b}$$

(b) *Two-component vector notation*

An obvious way to generalize one's mathematics from a single mode to two modes is to replace the single-mode annihilation and creation operators a and $a^\dagger$ by the two-component operator column vectors [29, 1]

$$\boldsymbol{a} \equiv \begin{pmatrix} a_+ \\ a_- \end{pmatrix}, \qquad \boldsymbol{a}^* \equiv \begin{pmatrix} a_+{}^\dagger \\ a_-{}^\dagger \end{pmatrix}. \tag{3.1.6}$$

The column vectors for the dimensionless position and momentum variables $\hat{x}_\pm$ and $\hat{p}_\pm$, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$, are related to $\boldsymbol{a}$ and $\boldsymbol{a}^*$ by

$$\hat{\boldsymbol{x}} \equiv \begin{pmatrix} \hat{x}_+ \\ \hat{x}_- \end{pmatrix} \equiv 2^{-1/2}(\boldsymbol{a} + \boldsymbol{a}^*)\,, \qquad \hat{\boldsymbol{p}} \equiv \begin{pmatrix} \hat{p}_+ \\ \hat{p}_- \end{pmatrix} \equiv 2^{-1/2}(-\mathrm{i}\boldsymbol{a} + \mathrm{i}\boldsymbol{a}^*)\,; \tag{3.1.7a}$$

$$\boldsymbol{a} \equiv 2^{-1/2}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{p}}) \tag{3.1.7b}$$

[cf. eqs. (2.1.1)]. The adjoints and transposes of these column vectors are defined in the usual way (see subsection 2.3). Similar definitions hold for column vectors of complex numbers; e.g., the column vector for the complex amplitudes μ_+ and μ_- is

$$\boldsymbol{\mu} \equiv \langle \boldsymbol{a} \rangle \equiv \begin{pmatrix} \mu_+ \\ \mu_- \end{pmatrix} \equiv 2^{-1/2}(\boldsymbol{x}_0 + \mathrm{i}\boldsymbol{p}_0)\,, \tag{3.1.8a}$$

where $\boldsymbol{x}_0$ and $\boldsymbol{p}_0$ are the column vectors for the mean positions $x_{0\pm}$ and momentums $p_{0\pm}$, respectively,

$$\boldsymbol{x}_0 \equiv \langle \hat{\boldsymbol{x}} \rangle \equiv \begin{pmatrix} x_{0+} \\ x_{0-} \end{pmatrix} = 2^{-1/2}(\boldsymbol{\mu} + \boldsymbol{\mu}^*), \tag{3.1.8b}$$

$$\boldsymbol{p}_0 \equiv \langle \hat{\boldsymbol{p}} \rangle \equiv \begin{pmatrix} p_{0+} \\ p_{0-} \end{pmatrix} = 2^{-1/2}(-\mathrm{i}\boldsymbol{\mu} + \mathrm{i}\boldsymbol{\mu}^*) \tag{3.1.8c}$$

[cf. eq. 2.1.3)]. This two-component vector notation is used throughout this section in order to present the two-mode results in a simple form that resembles as closely as possible the single-mode results. For example, the commutation relations (3.1.2) take on the matrix form

$$[\boldsymbol{a}, \boldsymbol{a}^\dagger] \equiv \boldsymbol{a}\boldsymbol{a}^\dagger - (\boldsymbol{a}^*\boldsymbol{a}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{1}, \qquad [\boldsymbol{a}, \boldsymbol{a}^{\mathrm{T}}] \equiv \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} - (\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})^{\mathrm{T}} = [a_+, a_-]\,\mathrm{i}\sigma_2 = 0\,; \tag{3.1.9a}$$

$$[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}^{\mathrm{T}}] \equiv \hat{\boldsymbol{x}}\hat{\boldsymbol{p}}^{\mathrm{T}} - (\hat{\boldsymbol{p}}\hat{\boldsymbol{x}}^{\mathrm{T}})^{\mathrm{T}} = \mathrm{i}\mathbf{1},$$

$$[\hat{\boldsymbol{x}}, \hat{\boldsymbol{x}}^{\mathrm{T}}] = [\hat{\boldsymbol{p}}, \hat{\boldsymbol{p}}^{\mathrm{T}}] = 0\,. \tag{3.1.9b}$$

(c) *Second-order noise moments*

For a single mode there are only two relevant Hermitian operators, $\hat{x}$ and $\hat{p}$ (or one complex operator, a), and hence only three real second-order noise moments to consider – $\langle(\Delta\hat{x})^2\rangle$, $\langle(\Delta\hat{p})^2\rangle$, and $\langle\Delta\hat{x}\,\Delta\hat{p}\rangle_{\mathrm{sym}}$, or, equivalently, $\langle(\Delta a)^2\rangle$ and $\langle|\Delta a|^2\rangle$. For two modes, there are four relevant Hermitian operators, $\hat{x}_\pm$ and $\hat{p}_\pm$, and hence ten real second-order noise moments to consider; six of these are associated with each of the modes separately, and four describe correlations between the modes. The four correlated noise moments are $\langle\Delta\hat{x}_+\,\Delta\hat{x}_-\rangle$, $\langle\Delta\hat{p}_+\,\Delta\hat{p}_-\rangle$, and $\langle\Delta\hat{x}_\pm\,\Delta\hat{p}_\mp\rangle$, or, equivalently, the complex noise moments $\langle\Delta a_+\,\Delta a_-\rangle$ and $\langle\Delta a_+\,\Delta a_-^\dagger\rangle$. The two-mode analog of the complex number $\langle(\Delta a)^2\rangle$ is the complex, symmetric matrix T:

$$T \equiv \langle\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^{\mathrm{T}}\rangle = \begin{pmatrix} \langle(\Delta a_+)^2\rangle & \langle\Delta a_+\,\Delta a_-\rangle \\ \langle\Delta a_+\,\Delta a_-\rangle & \langle(\Delta a_-)^2\rangle \end{pmatrix} = T^{\mathrm{T}}\,. \tag{3.1.10a}$$

The two-mode analog of the positive real number (mean-square uncertainty) $\langle|\Delta a|^2\rangle$ is the Hermitian matrix Q:

$$Q \equiv \langle\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^\dagger\rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}\langle\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^\dagger\rangle + \tfrac{1}{2}\langle\Delta\boldsymbol{a}^*\,\Delta\boldsymbol{a}^{\mathrm{T}}\rangle^{\mathrm{T}}$$

$$= \begin{pmatrix} \langle|\Delta a_+|^2\rangle & \langle\Delta a_+\,\Delta a_-^\dagger\rangle \\ \langle\Delta a_-\,\Delta a_+^\dagger\rangle & \langle|\Delta a_-|^2\rangle \end{pmatrix} = Q^\dagger \tag{3.1.10b}$$

[cf. eqs. (2.1.4)].

The two-mode analogs of the three real second-order noise moments of $\hat{x}$ and $\hat{p}$ are the three real, two-dimensional covariance matrices S_x, S_p, and S_{xp}:

$$S_x \equiv \langle\Delta\hat{\boldsymbol{x}}\,\Delta\hat{\boldsymbol{x}}^{\mathrm{T}}\rangle = \begin{pmatrix} \langle(\Delta\hat{x}_+)^2\rangle & \langle\Delta\hat{x}_+\,\Delta\hat{x}_-\rangle \\ \langle\Delta\hat{x}_+\,\Delta\hat{x}_-\rangle & \langle(\Delta\hat{x}_-)^2\rangle \end{pmatrix} = S_x^{\mathrm{T}}, \tag{3.1.11a}$$

$$S_p \equiv \langle\Delta\hat{\boldsymbol{p}}\,\Delta\hat{\boldsymbol{p}}^{\mathrm{T}}\rangle = \begin{pmatrix} \langle(\Delta\hat{p}_+)^2\rangle & \langle\Delta\hat{p}_+\,\Delta\hat{p}_-\rangle \\ \langle\Delta\hat{p}_+\,\Delta\hat{p}_-\rangle & \langle(\Delta\hat{p}_-)^2\rangle \end{pmatrix} = S_p^{\mathrm{T}}, \tag{3.1.11b}$$

$$S_{xp} \equiv \langle \Delta \hat{\boldsymbol{x}}\, \Delta \hat{\boldsymbol{p}}^{\mathrm{T}} \rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}\langle \Delta \hat{\boldsymbol{x}}\, \Delta \hat{\boldsymbol{p}}^{\mathrm{T}} \rangle + \tfrac{1}{2}\langle \Delta \hat{\boldsymbol{p}}\, \Delta \hat{\boldsymbol{x}}^{\mathrm{T}} \rangle^{\mathrm{T}}$$

$$= \begin{pmatrix} \langle \Delta \hat{x}_+\, \Delta \hat{p}_+ \rangle_{\mathrm{sym}} & \langle \Delta \hat{x}_+\, \Delta \hat{p}_- \rangle \\ \langle \Delta \hat{x}_-\, \Delta \hat{p}_+ \rangle & \langle \Delta \hat{x}_-\, \Delta \hat{p}_- \rangle_{\mathrm{sym}} \end{pmatrix}. \tag{3.1.11c}$$

The matrices S_x and S_p are positive semi-definite – i.e., their traces and determinants are nonnegative [47]; they are positive definite, if one excludes eigenstates of $\hat{x}_+$ and $\hat{x}_-$ or $\hat{p}_+$ and $\hat{p}_-$. Such states are excluded here, for although they can be viewed in a formal sense as limiting cases of GPS, they are not normalizable, since their wave functions are delta functions. Throughout this paper, therefore, S_x and S_p are positive definite.

The noise matrices T and Q are related to the covariance matrices S_x, S_p, and S_{xp} by

$$T = \tfrac{1}{2}(S_x - S_p) + \tfrac{1}{2}\mathrm{i}(S_{xp}{}^{\mathrm{T}} + S_{xp}), \tag{3.1.12a}$$

$$Q = \tfrac{1}{2}(S_x + S_p) + \tfrac{1}{2}\mathrm{i}(S_{xp}{}^{\mathrm{T}} - S_{xp}) \tag{3.1.12b}$$

[eqs. (3.1.1); cf. eqs. (2.1.5)]. These matrix equalities are a compact way of writing the following relations between the second-order noise moments:

$$\langle (\Delta a_\pm)^2 \rangle = \tfrac{1}{2}[\langle (\Delta \hat{x}_\pm)^2 \rangle - \langle (\Delta \hat{p}_\pm)^2 \rangle + 2\mathrm{i}\langle \Delta \hat{x}_\pm\, \Delta \hat{p}_\pm \rangle_{\mathrm{sym}}], \tag{3.1.13a}$$

$$\langle |\Delta a_\pm|^2 \rangle = \tfrac{1}{2}[\langle (\Delta \hat{x}_\pm)^2 \rangle + \langle (\Delta \hat{p}_\pm)^2 \rangle], \tag{3.1.13b}$$

$$\langle \Delta a_+\, \Delta a_- \rangle = \tfrac{1}{2}[\langle \Delta \hat{x}_+\, \Delta \hat{x}_- \rangle - \langle \Delta \hat{p}_+\, \Delta \hat{p}_- \rangle + \mathrm{i}(\langle \Delta \hat{x}_-\, \Delta \hat{p}_+ \rangle + \langle \Delta \hat{x}_+\, \Delta \hat{p}_- \rangle)], \tag{3.1.13c}$$

$$\langle \Delta a_+\, \Delta a_-{}^{\dagger} \rangle = \tfrac{1}{2}[\langle \Delta \hat{x}_+\, \Delta \hat{x}_- \rangle + \langle \Delta \hat{p}_+\, \Delta \hat{p}_- \rangle + \mathrm{i}(\langle \Delta \hat{x}_-\, \Delta \hat{p}_+ \rangle - \langle \Delta \hat{x}_+\, \Delta \hat{p}_- \rangle)]. \tag{3.1.13d}$$

The total noise of a two-mode GPS is the sum of the mean-square uncertainties,

$$\langle |\Delta a_+|^2 \rangle + \langle |\Delta a_-|^2 \rangle = \mathrm{Tr}\, Q = \tfrac{1}{2}\mathrm{Tr}(S_x + S_p). \tag{3.1.14}$$

The analog for two-mode states of the inequalities (2.1.6) is a matrix relation between the noise matrices S_x, S_p, and S_{xp}, or equivalently, between Q and T [49]. This relation says that the real matrix

$$S_x S_p - \tfrac{1}{4}\mathbf{1} - S_{xp}{}^2 \tag{3.1.15a}$$

and the Hermitian matrix

$$Q^2 - \tfrac{1}{4}\mathbf{1} - TT^* \tag{3.1.15b}$$

are both positive semi-definite (psd), and they vanish identically if and only if the state is an eigenstate of two independent kinds of linear combinations of $\hat{x}_+$, $\hat{x}_-$, $\hat{p}_+$, and $\hat{p}_-$ (or a_+, a_-, $a_+{}^{\dagger}$, and $a_-{}^{\dagger}$) – i.e., if and only if the state is a two-mode Gaussian pure state (see subsection 3.2). Hence for a two-mode GPS only six of the ten real second-order noise moments are independent; i.e., there are six independent real parameters associated with the second-order noise moments. Note that the diagonal elements of the matrices (3.1.15a) and (3.1.15b) are identical. A psd two-dimensional Hermitian matrix must have nonnegative diagonal elements, and it is equal to the null matrix if and only if both diagonal elements vanish. This implies the following two equivalent pairs of inequalities:

$$\langle|\Delta a_\pm|^2\rangle^2 \geq \tfrac{1}{4} + |\langle(\Delta a_\pm)^2\rangle|^2 + |\langle\Delta a_+\,\Delta a_-\rangle|^2 - |\langle\Delta a_+\,\Delta a_-^{\dagger}\rangle|^2\,, \tag{3.1.16a}$$

$$\langle(\Delta\hat{x}_\pm)^2\rangle\langle(\Delta\hat{p}_\pm)^2\rangle \geq \tfrac{1}{4} + \langle\Delta\hat{x}_\pm\,\Delta\hat{p}_\pm\rangle^2_{\text{sym}} + \langle\Delta\hat{x}_+\,\Delta\hat{p}_-\rangle\langle\Delta\hat{x}_-\,\Delta\hat{p}_+\rangle - \langle\Delta\hat{x}_+\,\Delta\hat{x}_-\rangle\langle\Delta\hat{p}_+\,\Delta\hat{p}_-\rangle\,. \tag{3.1.16b}$$

Equations (3.1.16a) and (3.1.16b) each represent a pair of inequalities (uncertainty principles), one for the upper ("+") sign, and one for the lower ("−") sign; the two pairs (not the members of each pair) are equivalent. Equalities hold in these expressions if and only if the matrices (3.1.15) vanish, i.e., if and only if the state is a two-mode Gaussian pure state. If the two modes are uncorrelated, these expressions reduce to the single-mode uncertainty principles (2.1.6) for each mode.

3.1.2. Two-mode rotation operators

Associated with each of the two modes is a single-mode rotation operator,

$$R_\pm(\theta) \equiv \exp(-\mathrm{i}\theta a_\pm^{\dagger} a_\pm)\,, \tag{3.1.17}$$

whose properties were discussed in subsection 2.1.2. It is useful to define another pair of commuting unitary operators, $R_s(\theta)$ and $R_a(\theta)$, equivalent to $R_+(\theta)$ and $R_-(\theta)$, as follows:

$$R_s(\theta) \equiv R_+(\theta)R_-(\theta) = \exp(-\mathrm{i}\theta \boldsymbol{a}^{\dagger}\boldsymbol{a})\,; \tag{3.1.18a}$$

$$R_a(\theta) \equiv R_+(\theta)R_-^{\dagger}(\theta) = \exp(-\mathrm{i}\theta \boldsymbol{a}^{\dagger}\sigma_3\boldsymbol{a})\,. \tag{3.1.18b}$$

In this paper the operators $R_s(\theta)$ and $R_a(\theta)$ are referred to as the symmetric and antisymmetric (two-mode) rotation operators, respectively. For notational convenience, denote the general product of two single-mode rotation operators (angles θ_+, θ_-) by the symbol $\mathbf{R}(\boldsymbol{\theta})$, and define angles θ_s and θ_a by

$$\mathbf{R}(\boldsymbol{\theta}) \equiv R_+(\theta_+)R_-(\theta_-) = R_s(\theta_s)R_a(\theta_a) = \exp[-\mathrm{i}\boldsymbol{a}^{\dagger}\boldsymbol{\theta}\boldsymbol{a}]\,, \tag{3.1.19a}$$

$$\boldsymbol{\theta} \equiv \theta_s\mathbf{1} + \theta_a\sigma_3 = \begin{pmatrix}\theta_+ & 0\\ 0 & \theta_-\end{pmatrix}, \tag{3.1.19b}$$

$$\theta_s \equiv \tfrac{1}{2}(\theta_+ + \theta_-)\,, \qquad \theta_a \equiv \tfrac{1}{2}(\theta_+ - \theta_-)\,. \tag{3.1.19c}$$

For two oscillators characterized by frequencies $\Omega \pm \varepsilon$, $R_s(\Omega t)R_a(\varepsilon t)$ is the evolution operator associated with the free Hamiltonian $H_0^{(2)}$ [50],

$$H_0^{(2)} = \Omega\boldsymbol{a}^{\dagger}\boldsymbol{a} + \varepsilon\boldsymbol{a}^{\dagger}\sigma_3\boldsymbol{a}\,, \tag{3.1.20a}$$

$$\exp[-\mathrm{i}H_0^{(2)}t] = R_s(\Omega t)R_a(\varepsilon t)\,. \tag{3.1.20b}$$

The rotation operators $R_s(\theta)$ and $R_a(\theta)$ acting on any (two-mode) number eigenstate $|n_+, n_-\rangle$ multiply it by the phase factors $\exp[-i(n_+ + n_-)\theta]$ and $\exp[-i(n_+ - n_-)\theta]$, respectively; in particular, they leave the vacuum state unchanged:

$$R_s(\theta)|0\rangle = |0\rangle\,, \qquad R_a(\theta)|0\rangle = |0\rangle\,. \tag{3.1.21}$$

A unitary transformation generated by the symmetric rotation operator $R_s(\theta)$ produces a common phase change of the annihilation operators – i.e., it transforms $\boldsymbol{a}$ into $e^{i\theta}\boldsymbol{a}$, and rotates $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ into each other:

$$R_s(\theta)\boldsymbol{a}R_s^\dagger(\theta) = e^{i\theta}\boldsymbol{a} = \begin{pmatrix} a_+(\theta) \\ a_-(\theta) \end{pmatrix}, \tag{3.1.22a}$$

$$R_s(\theta)\hat{\boldsymbol{x}}R_s^\dagger(\theta) = \hat{\boldsymbol{x}}\cos\theta - \hat{\boldsymbol{p}}\sin\theta = \begin{pmatrix} \hat{x}_+(\theta) \\ \hat{x}_-(\theta) \end{pmatrix}, \tag{3.1.22b}$$

$$R_s(\theta)\hat{\boldsymbol{p}}R_s^\dagger(\theta) = \hat{\boldsymbol{x}}\sin\theta + \hat{\boldsymbol{p}}\sin\theta = \begin{pmatrix} \hat{p}_+(\theta) \\ \hat{p}_-(\theta) \end{pmatrix} \tag{3.1.22c}$$

[eqs. (2.1.10)]. The notation here means that each component of the column vector undergoes the (same) unitary transformation. A unitary transformation generated by the antisymmetric rotation operator $R_a(\theta)$ produces an opposite phase change of the annihilation operators:

$$R_a(\theta)\boldsymbol{a}R_a^\dagger(\theta) = \exp(i\theta\sigma_3)\boldsymbol{a} = \begin{pmatrix} a_+(\theta) \\ a_-(-\theta) \end{pmatrix}, \tag{3.1.23a}$$

$$R_a(\theta)\hat{\boldsymbol{x}}R_a^\dagger(\theta) = \hat{\boldsymbol{x}}\cos\theta - \sigma_3\hat{\boldsymbol{p}}\sin\theta = \begin{pmatrix} \hat{x}_+(\theta) \\ \hat{x}_-(-\theta) \end{pmatrix}, \tag{3.1.23b}$$

$$R_a(\theta)\hat{\boldsymbol{p}}R_a^\dagger(\theta) = \sigma_3\hat{\boldsymbol{x}}\sin\theta + \hat{\boldsymbol{p}}\cos\theta = \begin{pmatrix} \hat{p}_+(\theta) \\ \hat{p}_-(-\theta) \end{pmatrix}. \tag{3.1.23c}$$

The product of two single-mode rotation operators therefore unitarily transforms $\boldsymbol{a}$ in the following way:

$$\mathbf{R}(\boldsymbol{\theta})\boldsymbol{a}\mathbf{R}^\dagger(\boldsymbol{\theta}) = \begin{pmatrix} a_+(\theta_+) \\ a_-(\theta_-) \end{pmatrix} = \exp(i\theta_s)\exp(i\theta_a\sigma_3)\boldsymbol{a} = \exp(i\boldsymbol{\theta})\boldsymbol{a} \equiv \boldsymbol{a}(\boldsymbol{\theta})\,; \tag{3.1.24a}$$

$$\mathbf{R}(\boldsymbol{\theta})\hat{\boldsymbol{x}}\mathbf{R}^\dagger(\boldsymbol{\theta}) = \begin{pmatrix} \hat{x}_+(\theta_+) \\ \hat{x}_-(\theta_-) \end{pmatrix} \equiv \hat{\boldsymbol{x}}(\boldsymbol{\theta})\,; \tag{3.1.24b}$$

$$\mathbf{R}(\boldsymbol{\theta})\hat{\boldsymbol{p}}\mathbf{R}^\dagger(\boldsymbol{\theta}) = \begin{pmatrix} \hat{p}_+(\theta_+) \\ \hat{p}_-(\theta_-) \end{pmatrix} \equiv \hat{\boldsymbol{p}}(\boldsymbol{\theta}) \tag{3.1.24c}$$

[cf. eqs. (2.1.10)].

Both $R_s(\theta)$ and $R_a(\theta)$ preserve the total number of photons in each mode separately, i.e., they preserve both the sum and the difference of the number of photons:

$$\langle R_s^\dagger(\theta)\boldsymbol{a}^\dagger\boldsymbol{a}R_s(\theta)\rangle = \langle \boldsymbol{a}^\dagger\boldsymbol{a}\rangle\,, \qquad \langle R_a^\dagger(\theta)\boldsymbol{a}^\dagger\boldsymbol{a}R_a(\theta)\rangle = \langle \boldsymbol{a}^\dagger\boldsymbol{a}\rangle\,; \tag{3.1.25a}$$

$$\langle R_s^\dagger(\theta)\boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}R_s(\theta)\rangle = \langle \boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}\rangle\,, \qquad \langle R_a^\dagger(\theta)\boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}R_a(\theta)\rangle = \langle \boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}\rangle\,. \tag{3.1.25b}$$

They therefore also preserve the total noise of each mode separately. This is seen by replacing $\boldsymbol{a}^\dagger\boldsymbol{a}$ in eq. (3.1.25a) by the operator for the sum of the total noises,

$$(\Delta\boldsymbol{a}^\dagger\,\Delta\boldsymbol{a})_{\text{sym}} \equiv \tfrac{1}{2}(\Delta\boldsymbol{a}^\dagger\,\Delta\boldsymbol{a} + \Delta\boldsymbol{a}^{\mathrm{T}}\,\Delta\boldsymbol{a}^*) = |\Delta a_+|^2 + |\Delta a_-|^2\,, \tag{3.1.26}$$

and replacing $\boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}$ in eq. (3.1.25b) by the operator for the difference in the total noises of the two modes,

$$\Delta\boldsymbol{a}^\dagger\sigma_3\,\Delta\boldsymbol{a} = \Delta\boldsymbol{a}^{\mathrm{T}}\sigma_3\,\Delta\boldsymbol{a}^* = |\Delta a_+|^2 - |\Delta a_-|^2\,. \tag{3.1.27}$$

[Recall that the operators $\Delta a_\pm$ are defined only with reference to a particular state, which defines $\langle a_\pm\rangle$; see comment after eq. (2.1.12b).] The noise matrices T and Q for a state $|\Psi\rangle$ are related to those of the rotated state $\mathbf{R}(\boldsymbol{\theta})|\Psi\rangle$, $T(-\boldsymbol{\theta})$ and $Q(-\boldsymbol{\theta})$, respectively, as follows:

$$\begin{aligned}\langle \mathbf{R}^\dagger(\boldsymbol{\theta})\,\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^{\mathrm{T}}\mathbf{R}(\boldsymbol{\theta})\rangle &= \langle \Delta\boldsymbol{a}(-\boldsymbol{\theta})\,\Delta\boldsymbol{a}^{\mathrm{T}}(-\boldsymbol{\theta})\rangle = \exp(-\mathrm{i}\boldsymbol{\theta})T\exp(-\mathrm{i}\boldsymbol{\theta}) \equiv T(-\boldsymbol{\theta})\\ &= \begin{pmatrix} \exp(-2\mathrm{i}\theta_+)\langle(\Delta a_+)^2\rangle & \exp(-2\mathrm{i}\theta_s)\langle\Delta a_+\,\Delta a_-\rangle \\ \exp(-2\mathrm{i}\theta_s)\langle\Delta a_+\,\Delta a_-\rangle & \exp(-2\mathrm{i}\theta_-)\langle(\Delta a_-)^2\rangle\end{pmatrix}\end{aligned} \tag{3.1.28a}$$

$$\begin{aligned}\langle \mathbf{R}^\dagger(\boldsymbol{\theta})(\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^\dagger)_{\text{sym}}\mathbf{R}(\boldsymbol{\theta})\rangle &= \langle \Delta\boldsymbol{a}(-\boldsymbol{\theta})\,\Delta\boldsymbol{a}^\dagger(-\boldsymbol{\theta})\rangle_{\text{sym}} = \exp(-\mathrm{i}\boldsymbol{\theta})Q\exp(\mathrm{i}\boldsymbol{\theta}) \equiv Q(-\boldsymbol{\theta})\\ &= \begin{pmatrix} \langle|\Delta a_+|^2\rangle & \exp(-2\mathrm{i}\theta_a)\langle\Delta a_+\,\Delta a_-^\dagger\rangle \\ \exp(2\mathrm{i}\theta_a)\langle\Delta a_-\,\Delta a_+^\dagger\rangle & \langle|\Delta a_-|^2\rangle\end{pmatrix}\end{aligned} \tag{3.1.28b}$$

[eq. (3.1.24a); cf. eqs. (2.1.12)].

3.1.3. Two-mode displacement operator

The two-mode displacement operator [10, 11] is simply a product of two single-mode displacement operators,

$$\begin{aligned}D(\boldsymbol{a},\boldsymbol{\mu}) &\equiv D(a_+,\mu_+)D(a_-,\mu_-) = \exp[\boldsymbol{a}^\dagger\boldsymbol{\mu} - \boldsymbol{\mu}^\dagger\boldsymbol{a}]\\ &= \exp[\mathrm{i}(\boldsymbol{p}_0^{\mathrm{T}}\hat{\boldsymbol{x}} - \boldsymbol{x}_0^{\mathrm{T}}\hat{\boldsymbol{p}})] = \exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}_0^{\mathrm{T}}\boldsymbol{x}_0)\exp[\mathrm{i}\boldsymbol{p}_0^{\mathrm{T}}\hat{\boldsymbol{x}} - \mathrm{i}\boldsymbol{x}_0^{\mathrm{T}}\hat{\boldsymbol{p}}]\end{aligned} \tag{3.1.29}$$

[cf. eq. (2.1.14)]. It satisfies the following equalities:

$$D^{-1}(\boldsymbol{a},\boldsymbol{\mu}) = D^\dagger(\boldsymbol{a},\boldsymbol{\mu}) = D(\boldsymbol{a},-\boldsymbol{\mu}) = D(-\boldsymbol{a},\boldsymbol{\mu})\,. \tag{3.1.30}$$

The properties of $D(\boldsymbol{a}, \boldsymbol{\mu})$ follow directly from those of the single-mode displacement operator $D(a, \mu)$ (subsection 2.1.3). Most important is the way it unitarily transforms the annihilation operators for the two modes:

$$D(\boldsymbol{a}, \boldsymbol{\mu})\boldsymbol{a}D^{\dagger}(\boldsymbol{a}, \boldsymbol{\mu}) = \boldsymbol{a} - \boldsymbol{\mu}\,. \tag{3.1.31}$$

This implies that when the displacement operator acts on a (two-mode) state, it preserves all noise moments of $a_{\pm}$ and $a_{\pm}{}^{\dagger}$.

Two other properties of the two-mode displacement operator are useful here. They are the two-mode analogs of the properties (2.1.17)–(2.1.20). First, it is unitarily transformed by the product of two single-mode rotation operators in the following way:

$$\begin{aligned}\mathbf{R}(\boldsymbol{\theta})D(\boldsymbol{a}, \boldsymbol{\mu})\mathbf{R}(\boldsymbol{\theta})^{\dagger} &= D[\boldsymbol{a}(\boldsymbol{\theta}), \boldsymbol{\mu}] = D[\boldsymbol{a}, \boldsymbol{\mu}(-\boldsymbol{\theta})] = D(\boldsymbol{a}, \exp(-\mathrm{i}\boldsymbol{\theta})\boldsymbol{\mu}) \\ &= D[a_+, \mu_+(-\theta_+)]D[a_-, \mu_-(-\theta_-)]\,,\end{aligned} \tag{3.1.32a}$$

$$\boldsymbol{\mu}(\boldsymbol{\theta}) \equiv \exp(\mathrm{i}\boldsymbol{\theta})\boldsymbol{\mu} \equiv \exp(\mathrm{i}\theta_s)\exp(\mathrm{i}\theta_a\sigma_3)\boldsymbol{\mu} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_+}\mu_+ \\ \mathrm{e}^{\mathrm{i}\theta_-}\mu_- \end{pmatrix} \equiv \begin{pmatrix} \mu_+(\theta_+) \\ \mu_-(\theta_-) \end{pmatrix} \tag{3.1.32b}$$

[eqs. (3.1.24a), (3.1.29)]. This transformation shows that the form of the displacement operator is invariant under unitary transformations of a_+ and a_- generated by the rotation operators:

$$D(\boldsymbol{a}, \boldsymbol{\mu}) = D[\boldsymbol{a}(\boldsymbol{\theta}), \boldsymbol{\mu}(\boldsymbol{\theta})]\,. \tag{3.1.33}$$

Second, the product of two two-mode displacement operators is another two-mode displacement operator, multiplied by a phase factor:

$$D(\boldsymbol{a}, \boldsymbol{\mu}')D(\boldsymbol{a}, \boldsymbol{\mu}) = \exp[\mathrm{i}\,\mathrm{Im}(\boldsymbol{\mu}'^{\dagger}\boldsymbol{\mu})]D(\boldsymbol{a}, \boldsymbol{\mu} + \boldsymbol{\mu}') \tag{3.1.34}$$

[cf. eq. (2.1.19)]. These properties, like the transformations (3.1.24a) and (3.1.31), show that any eigenstate of a_+ or a_- remains an eigenstate of a_+ or a_- when displaced and/or allowed to evolve freely. A two-mode coherent state [eq. (1.15)], for example, changes in the following way as it evolves freely:

$$\mathbf{R}(\boldsymbol{\theta})|\boldsymbol{\mu}\rangle_{\mathrm{coh}} = |\boldsymbol{\mu}(-\boldsymbol{\theta})\rangle_{\mathrm{coh}} \equiv |\mathrm{e}^{-\mathrm{i}\theta_+}\mu_+, \mathrm{e}^{-\mathrm{i}\theta_-}\mu_-\rangle_{\mathrm{coh}}\,, \tag{3.1.35}$$

where $\theta_{\pm} \equiv (\Omega \pm \varepsilon)t$ [cf. eq. (2.1.20)].

3.1.4. Mixing operator

The two-mode mixing operator $T(q, \chi)$ is defined by

$$T(q, \chi) \equiv \exp[q(\mathrm{e}^{-2\mathrm{i}\chi}a_-{}^{\dagger}a_+ - \mathrm{e}^{2\mathrm{i}\chi}a_+{}^{\dagger}a_-)] = \exp[-\mathrm{i}q\boldsymbol{a}^{\dagger}\sigma_{\chi-\pi/4}\boldsymbol{a}]\,, \tag{3.1.36a}$$

$$0 \le q \le \tfrac{1}{2}\pi\,, \qquad -\tfrac{1}{2}\pi < \chi \le \tfrac{1}{2}\pi \tag{3.1.36b}$$

[eqs. (1.10), (2.3.6b)]. It satisfies the following equalities:

$$T^{-1}(q,\chi)=T^{\dagger}(q,\chi)=T(-q,\chi)=T(q,\chi+\tfrac{1}{2}\pi)\,. \tag{3.1.37}$$

It is called a mixing operator because it unitarily transforms the annihilation operators for the two modes into each other:

$$T(q,\chi)\boldsymbol{a}T^{\dagger}(q,\chi)=F_{q,\chi}\boldsymbol{a}\,, \tag{3.1.38a}$$

$$F_{q,\chi}\equiv\begin{pmatrix}\cos q & e^{2i\chi}\sin q\\ -e^{-2i\chi}\sin q & \cos q\end{pmatrix}=\cos q\mathbf{1}+i\sin q\,\sigma_{\chi-\pi/4}=\exp(iq\sigma_{\chi-\pi/4})\,. \tag{3.1.38b}$$

[Note that the matrix $\sigma_{\chi-\pi/4}$ appears in both $T(q,\chi)$ and the transformation matrix $F_{q,\chi}$ because the matrix of commutators $[\boldsymbol{a},\boldsymbol{a}^{\dagger}]=\mathbf{1}$ is the identity matrix, i.e., because $[\boldsymbol{a}^{\dagger}K\boldsymbol{a},\boldsymbol{a}]=-K\boldsymbol{a}$; cf. eqs. (2.3.10).] The unitary matrix $F_{q,\chi}$ has the following important properties:

$$F_{q,\chi}{}^{-1}=F_{q,\chi}{}^{\dagger}=F_{-q,\chi}=F_{q,\chi+\pi/2}=\sigma_3F_{q,\chi}\sigma_3\,, \tag{3.1.39a}$$

$$F_{q,0}=\exp(iq\sigma_2)=\begin{pmatrix}\cos q & \sin q\\ -\sin q & \cos q\end{pmatrix},\qquad F_{q,\pi/4}=\exp(iq\sigma_1)=\begin{pmatrix}\cos q & i\sin q\\ i\sin q & \cos q\end{pmatrix}, \tag{3.1.39b}$$

$$F_{q,\chi}=\exp(i\chi\sigma_3)F_{q,\chi-\chi'}\exp(-i\chi'\sigma_3)\,, \tag{3.1.39c}$$

$$F_{q,\chi}F_{q',\chi}=F_{q+q',\chi}\,. \tag{3.1.39d}$$

The transformation (3.1.38a) ensures that states unitarily related to eigenstates of a_+ and a_- by mixing operators are themselves eigenstates of a_+ and a_-. This shows, for example, that the mixing operator, like the rotation operators, leaves the vacuum state unchanged:

$$T(q,\chi)|0\rangle=|0\rangle\,. \tag{3.1.40}$$

This can also be seen from the factored forms for $T(q,\chi)$ given below [eq. (3.1.45)].

The unitarity of $F_{q,\chi}$ ensures that the mixing operator preserves the total number of photons in the two modes:

$$\langle T^{\dagger}(q,\chi)\boldsymbol{a}^{\dagger}\boldsymbol{a}T(q,\chi)\rangle=\langle\boldsymbol{a}^{\dagger}F_{q,\chi}F_{q,\chi}{}^{\dagger}\boldsymbol{a}\rangle=\langle\boldsymbol{a}^{\dagger}\boldsymbol{a}\rangle\,. \tag{3.1.41a}$$

It therefore also preserves the total noise of the two modes:

$$\langle T^{\dagger}(q,\chi)(\Delta\boldsymbol{a}^{\dagger}\,\Delta\boldsymbol{a})_{\rm sym}T(q,\chi)\rangle=\langle\Delta\boldsymbol{a}^{\dagger}\,\Delta\boldsymbol{a}\rangle_{\rm sym}=\langle|\Delta a_+|^2\rangle+\langle|\Delta a_-|^2\rangle\,. \tag{3.1.41b}$$

The noise matrices T and Q for a state $|\Psi\rangle$ are related to those of the transformed state $T(q,\chi)|\Psi\rangle$ in the following ways:

$$\langle T^{\dagger}(q,\chi)\,\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^{\rm T}T(q,\chi)\rangle=F_{q,\chi}{}^{\dagger}TF_{q,\chi}{}^{*}\,, \tag{3.1.42a}$$

$$\langle T^{\dagger}(q,\chi)(\Delta\boldsymbol{a}\,\Delta\boldsymbol{a}^{\dagger})_{\rm sym}T(q,\chi)\rangle=F_{q,\chi}{}^{\dagger}QF_{q,\chi} \tag{3.1.42b}$$

[eqs. (3.1.38)]. For states whose noise matrix Q is proportional to the identity matrix **1**, the total noise of each mode separately is left unchanged when the state is multiplied by rotation or mixing operators [eqs. (3.1.28b), (3.1.42b)]. Such states include all eigenstates of a_+ and a_-, all two-mode squeezed states, and all products of two single-mode squeezed states with identical squeeze factors. For the general two-mode GPS (1.18), however, the total noise of each mode is separately preserved only when the state is multiplied by a symmetric rotation operators $R_s(\theta)$ or by a particular mixing operator, whose parameters q and χ satisfy

$$(\langle|\Delta a_+|^2\rangle - \langle|\Delta a_-|^2\rangle)\sin q - 2\,\mathrm{Re}(\mathrm{e}^{-2\mathrm{i}\chi}\langle\Delta a_+\,\Delta a_-{}^\dagger\rangle)\cos q = 0\,. \tag{3.1.43}$$

The mixing operator and the two rotation operators represent all the unitary operators that induce matrix transformations on the column vector $\boldsymbol{a}$. The transformation matrices associated with them comprise the group U(2) of two-dimensional, unitary matrices M_{U} that preserve the identity matrix, i.e., that preserve the commutator matrix $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \mathbf{1}$, $M_{\mathrm{U}}\mathbf{1}M_{\mathrm{U}}{}^\dagger = \mathbf{1}$. The most general element of this group has the form

$$\begin{aligned} M_{\mathrm{U}} &= \exp(\mathrm{i}\theta_{\mathrm{s}})\exp(\mathrm{i}\theta_{\mathrm{a}}\sigma_3)F_{q,\chi} = \exp(\mathrm{i}\theta_{\mathrm{s}})F_{q,\chi+\theta_{\mathrm{a}}}\exp(\mathrm{i}\theta_{\mathrm{a}}\sigma_3) \\ &= \exp(\mathrm{i}\theta_{\mathrm{s}})\begin{pmatrix} \exp(\mathrm{i}\theta_{\mathrm{a}})\cos a & \exp[\mathrm{i}(2\chi+\theta_{\mathrm{a}})]\sin q \\ -\exp[-\mathrm{i}(2\chi+\theta_{\mathrm{a}})]\sin q & \exp(-\mathrm{i}\theta_{\mathrm{a}})\cos q \end{pmatrix} \end{aligned} \tag{3.1.44a}$$

for real numbers θ_{s}, θ_{a}, q, and χ. The matrix (3.1.44a) is the transformation matrix that results from a unitary transformation of $\boldsymbol{a}$ by a product of the two rotation operators and a mixing operator, i.e.,

$$M_{\mathrm{U}}\boldsymbol{a} = T(q,\chi)\mathbf{R}(\boldsymbol{\theta})\boldsymbol{a}\mathbf{R}^\dagger(\boldsymbol{\theta})T^\dagger(q,\chi)\,. \tag{3.1.44b}$$

The unitary transformation matrices associated with the mixing operator $T(q,\chi)$ and the antisymmetric rotation operator $R_{\mathrm{a}}(\theta)$ form the three-parameter group SU(2), the elements of U(2) with unity determinant. The underlying Lie algebra for the group SU(2) is that of the operators $a_-{}^\dagger a_+$, $a_+{}^\dagger a_-$, and $(a_-{}^\dagger a_- - a_+{}^\dagger a_+)$, i.e., the generators of $T(q,\chi)$ and $R_{\mathrm{a}}(\theta)$.

Properties of the mixing operator can be obtained directly from properties of the matrix $F_{q,\chi}$, just as properties of the single-mode squeeze operator are obtained from properties of the transformation matrix $C_{r,\varphi}$ (see subsection 2.3). For example, the mixing operator can be factored into a product of exponentials of the operators $a_-{}^\dagger a_+$, $a_+{}^\dagger a_-$, and $(a_-{}^\dagger a_- - a_+{}^\dagger a_+)$, simply by factoring the matrix $F_{q,\chi}$ into exponentials of matrices (linear combinations of the Pauli matrices) that have the same commutator algebra as these operators. Examples of such matrices are σ_-, σ_+, and $-\sigma_3$, where $\sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$. The mixing operator $T(q,\chi)$ is thus found to have the following equivalent factored forms:

$$\begin{aligned} T(q,\chi) &= \exp(-\Lambda A^\dagger)\exp(\Lambda^*\,\mathrm{e}^{2f}A)\,\mathrm{e}^{fB} = \exp(\Lambda^*A)\exp(-\Lambda\,\mathrm{e}^{2f}A^\dagger)\,\mathrm{e}^{-fB} \\ &= \exp(-\Lambda A^\dagger)\,\mathrm{e}^{fB}\exp(\Lambda^*A) = \mathrm{e}^{fB}\exp(-\Lambda\,\mathrm{e}^{2f}A^\dagger)\exp(\Lambda^*A) \\ &= \mathrm{e}^{-fB}\exp(\Lambda^*\,\mathrm{e}^{2f}A)\exp(-\Lambda A^\dagger) = \exp(\Lambda^*A)\,\mathrm{e}^{-fB}\exp(-\Lambda A^\dagger)\,, \end{aligned}$$

$$\Lambda \equiv e^{2i\chi} \tan q\,, \qquad f \equiv \ln(\cos q)\,,$$
$$A \equiv a_-{}^{\dagger} a_+\,, \qquad B \equiv a_-{}^{\dagger} a_- - a_+{}^{\dagger} a_+\,. \tag{3.1.45}$$

The matrix equality

$$F_{q,\chi} F^{\dagger}_{q',\chi'} = F_{\bar{q},\bar{\chi}} \exp(i\theta\sigma_3) = \exp(i\theta\sigma_3) F_{\bar{q},\bar{\chi}-\theta} \tag{3.1.46a}$$

implies that the product of two different mixing operators is another mixing operator, multiplied by a rotation operator:

$$T^{\dagger}(q',\chi') T(q,\chi) = R_a(\theta) T(\bar{q},\bar{\chi}) = T(\bar{q},\bar{\chi}-\theta) R_a(\theta)\,; \tag{3.1.46b}$$

here the real numbers $\bar{q}$, $\bar{\chi}$, and θ are related to q, q', χ, and χ' by

$$e^{i\theta} \cos \bar{q} = \cos q \cos q' + \exp[2i(\chi - \chi')] \sin q \sin q'\,, \tag{3.1.46c}$$

$$\exp[-i(\theta - 2\bar{\chi})] \sin \bar{q} = e^{2i\chi} \sin q \cos q' - e^{2i\chi'} \sin q' \cos q\,. \tag{3.1.46d}$$

For the special case $\chi = \chi'$ this gives the simple relation

$$T(q,\chi) T(q',\chi) = T(q + q',\chi) \tag{3.1.46e}$$

[eq. (3.1.39d)].

The property (3.1.39c) of the transformation matrix $F_{q,\chi}$ implies that the mixing operator $T(q,\chi)$ is unitarily transformed by the rotation operators in the following way:

$$\mathbf{R}(\boldsymbol{\theta}) T(q,\chi) \mathbf{R}^{\dagger}(\boldsymbol{\theta}) = T(q,\chi - \theta_a)\,. \tag{3.1.47}$$

Thus, $T(q,\chi)$ commutes with the symmetric rotation operator $R_s(\theta)$ but not with the antisymmetric rotation operator $R_a(\theta)$. This reveals why it preserves the total number of photons (hence the total noise), but not the difference in the number of photons in the two modes. The mixing operator unitarily transforms the two-mode displacement operator in the following way:

$$T^{\dagger}(q,\chi) D(\boldsymbol{a},\boldsymbol{\mu}) T(q,\chi) = D(F_{q,\chi}{}^{\dagger}\boldsymbol{a},\boldsymbol{\mu}) = D(\boldsymbol{a}, F_{q,\chi}\boldsymbol{\mu})\,. \tag{3.1.48a}$$

The form of the (two-mode) displacement operator is thus invariant under unitary transformations of $\boldsymbol{a}$ generated by a mixing operator:

$$D(\boldsymbol{a},\boldsymbol{\mu}) = D(F_{q,\chi}\boldsymbol{a}, F_{q,\chi}\boldsymbol{\mu})\,. \tag{3.1.48b}$$

One can now verify the statement made in the Introduction: a state unitarily related to a two-mode coherent state by any product of rotation, displacement, and mixing operators is equal to another coherent state (multiplied by an unobservable phase factor). For example,

$$\mathbf{R}(\boldsymbol{\theta}) T(q,\chi) D(\boldsymbol{a},\boldsymbol{\mu}') |\boldsymbol{\mu}\rangle_{\mathrm{coh}} = \exp[i\,\mathrm{Im}(\boldsymbol{\mu}'^{\dagger}\boldsymbol{\mu})] |\bar{\boldsymbol{\mu}}\rangle_{\mathrm{coh}}\,,$$
$$\bar{\boldsymbol{\mu}} \equiv \exp(-i\boldsymbol{\theta}) F_{-q,\chi}(\boldsymbol{\mu} + \boldsymbol{\mu}')\,. \tag{3.1.49}$$

All two-mode states that are eigenstates of both a_+ and a_- are unitarily related to the vacuum state by products of rotation, displacement, and mixing operators. Conversely, all such states are eigenstates of a_+ and a_-. These states comprise the entire class of two-mode states whose total noise is equal to that of the vacuum state. The special properties of the rotation and mixing operators – that they preserve the total number of photons, that they preserve the total noise, and that they preserve coherent states – are a consequence of one essential property: these operators generate the most general unitary matrix transformation of the annihilation operators and nothing more; i.e., they never mix creation operators with annihilation operators. To find unitary operators that do not conserve the total noise and that generate new states from coherent states (states with a total noise greater than that of the vacuum state), one must consider operators – the single-mode and two-mode squeeze operators – that mix creation and annihilation operators.

3.1.5. Squeeze operators for two modes

(a) *Single-mode squeeze operators*

The single-mode squeeze operators

$$S_{1\pm}(r,\varphi) \equiv \exp[\tfrac{1}{2}r(\mathrm{e}^{-2\mathrm{i}\varphi}a_\pm{}^2 - \mathrm{e}^{2\mathrm{i}\varphi}a_\pm{}^{\dagger 2})] \tag{3.1.50}$$

were discussed in subsection 2.1.4. Each single-mode squeeze operator unitarily transforms the annihilation operator for its mode into a linear combination of the annihilation and creation operator for that mode:

$$S_{1\pm}(r,\varphi)a_\pm S_{1\pm}{}^\dagger(r,\varphi) = a_\pm \cosh r + a_\pm{}^\dagger\,\mathrm{e}^{2\mathrm{i}\varphi}\sinh r \tag{3.1.51}$$

[eqs. (2.1.21)–(2.1.23)]. In vector notation these transformations are

$$S_{1+}(r_+,\varphi_+)S_{1-}(r_-,\varphi_-)\boldsymbol{a}S_{1-}{}^\dagger(r_-,\varphi_-)S_{1+}{}^\dagger(r_+,\varphi_+) = P_{1\mathrm{c}}\boldsymbol{a} + P_{1\mathrm{s}}\boldsymbol{a}^*\,, \tag{3.1.52a}$$

$$P_{1\mathrm{c}} \equiv \begin{pmatrix} \cosh r_+ & 0 \\ 0 & \cosh r_- \end{pmatrix}, \qquad P_{1\mathrm{s}} \equiv \begin{pmatrix} \mathrm{e}^{2\mathrm{i}\varphi_+}\sinh r_+ & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\varphi_-}\sinh r_- \end{pmatrix}. \tag{3.1.52b}$$

Recall that a single-mode squeeze operator preserves neither the total number of photons nor the total noise of a mode [eqs. (2.1.24)].

(b) *Two-mode squeeze operator*

The two-mode squeeze operator $S(r,\varphi)$ is defined by

$$\begin{aligned} S(r,\varphi) &\equiv \exp[r(\mathrm{e}^{-2\mathrm{i}\varphi}a_+a_- - \mathrm{e}^{2\mathrm{i}\varphi}a_+{}^\dagger a_-{}^\dagger)] \\ &= \exp[\tfrac{1}{2}r(\mathrm{e}^{-2\mathrm{i}\varphi}\boldsymbol{a}^{\mathrm{T}}\sigma_1\boldsymbol{a} - \mathrm{e}^{2\mathrm{i}\varphi}\boldsymbol{a}^\dagger\sigma_1\boldsymbol{a}^*)]\,, \end{aligned} \tag{3.1.53a}$$

$$0 \le r < \infty\,, \qquad -\tfrac{1}{2}\pi < \varphi \le \tfrac{1}{2}\pi \tag{3.1.53b}$$

[eq. (1.13)]; it is denoted simply by S when lack of reference to a particular r and φ does not lead to confusion. It satisfies the following equalities:

$$S^{-1}(r,\varphi)=S^{\dagger}(r,\varphi)=S(-r,\varphi)=S(r,\varphi+\tfrac{1}{2}\pi)\,. \tag{3.1.54}$$

Properties of $S(r,\varphi)$ are discussed in refs. [19] and [20]. Most important is that it unitarily transforms $a_\pm$ into a linear combination of $a_\pm$ and $a_\mp{}^\dagger$:

$$S(r,\varphi)a_\pm S^\dagger(r,\varphi)=a_\pm\cosh r+a_\mp{}^\dagger\,e^{2i\varphi}\sinh r\equiv\alpha_\pm(r,\varphi) \tag{3.1.55a}$$

[eq. (1.20)]. In vector notation, these transformations take the form

$$S(r,\varphi)\boldsymbol{a}S^\dagger(r,\varphi)=\boldsymbol{a}\cosh r+\sigma_1\boldsymbol{a}^*\,e^{2i\varphi}\sinh r=\begin{pmatrix}\alpha_+(r,\varphi)\\ \alpha_-(r,\varphi)\end{pmatrix}\equiv\boldsymbol{\alpha}(r,\varphi)\,. \tag{3.1.55b}$$

A state unitarily related to an eigenstate of a_+ and a_- by a two-mode squeeze operator is an eigenstate of two "two-mode squeezed annihilation operators" [18–20] $\alpha_\pm(r,\varphi)$ (denoted simply by $\alpha_\pm$ or by the column vector $\boldsymbol{\alpha}$ when lack of reference to a particular r and φ does not lead to confusion). Inverting eq. (3.1.55b) gives the vector expression for $\boldsymbol{a}$ in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$:

$$\boldsymbol{a}=S^\dagger(r,\varphi)\boldsymbol{\alpha}(r,\varphi)S(r,\varphi)=\boldsymbol{\alpha}\cosh r-\sigma_1\boldsymbol{\alpha}^*\,e^{2i\varphi}\sinh r\,. \tag{3.1.56}$$

The unitarity of S ensures that the commutator algebra of $\alpha_\pm$ and $\alpha_\pm{}^\dagger$ is identical to that of $a_\pm$ and $a_\pm{}^\dagger$:

$$[\boldsymbol{\alpha},\boldsymbol{\alpha}^\dagger]=[\boldsymbol{a},\boldsymbol{a}^\dagger]=\mathbf{1}\,, \tag{3.1.57a}$$

$$[\boldsymbol{\alpha},\boldsymbol{\alpha}^{\mathrm{T}}]=[\alpha_+,\alpha_-]i\sigma_2=[\boldsymbol{a},\boldsymbol{a}^{\mathrm{T}}]=0 \tag{3.1.57b}$$

[eq. (3.1.9a)].

The transformations (3.1.55) show that the two-mode squeeze operator preserves neither the total number of photons nor the total noise of a two-mode state:

$$\langle S^\dagger\boldsymbol{a}^\dagger\boldsymbol{a}S\rangle=\sinh^2 r+\cosh 2r\langle\boldsymbol{a}^\dagger\boldsymbol{a}\rangle-2\sinh 2r\,\mathrm{Re}(e^{-2i\varphi}\langle a_+a_-\rangle)\,, \tag{3.1.58a}$$

$$\langle S^\dagger(\Delta\boldsymbol{a}^\dagger\,\Delta\boldsymbol{a})_{\mathrm{sym}}S\rangle=\cosh 2r\langle\Delta\boldsymbol{a}^\dagger\,\Delta\boldsymbol{a}\rangle_{\mathrm{sym}}-2\sinh 2r\,\mathrm{Re}(e^{-2i\varphi}\langle\Delta a_+\,\Delta a_-\rangle) \tag{3.1.58b}$$

[cf. eqs. (2.1.24)]. Equation (3.1.58b) shows explicitly that any (two-mode) state whose unitary relation to the vacuum state (or to any eigenstate of a_+ and a_-) includes a two-mode squeeze operator has a total noise greater than that of the vacuum state. The two-mode squeeze operator does, however, preserve the difference in the number of photons, and therefore also the difference in the total noises, of the two modes:

$$\langle S^\dagger\boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}S\rangle=\langle\boldsymbol{a}^\dagger\sigma_3\boldsymbol{a}\rangle\,, \tag{3.1.59a}$$

$$\langle S^\dagger\,\Delta\boldsymbol{a}^\dagger\sigma_3\,\Delta\boldsymbol{a}S\rangle=\langle\Delta\boldsymbol{a}^\dagger\sigma_3\,\Delta\boldsymbol{a}\rangle\,. \tag{3.1.59b}$$

The relations between the noise matrices T and Q for a state $|\Psi\rangle$ and for the state $S(r,\varphi)|\Psi\rangle$ follow straightforwardly from the transformation (3.1.55b). The noise matrices T and Q for a two-mode squeezed state are given explicitly below (see section 3.1.7).

A few other properties of the two-mode squeeze operator are useful here. First it is unitarily transformed by the rotation operators in the following way:

$$\mathbf{R}(\boldsymbol{\theta})S(r,\varphi)\mathbf{R}^{\dagger}(\boldsymbol{\theta}) = S(r,\varphi-\theta_{\mathrm{s}}) \tag{3.1.60}$$

[eqs. (2.1.24a) and (3.1.53) or (2.3.14c); cf. eqs. (2.1.25) or (2.3.19b)]. Thus, $S(r,\varphi)$ commutes with $R_{\mathrm{a}}(\theta)$ but not with $R_{\mathrm{s}}(\theta)$. This reveals why it preserves the difference in the number of photons (hence the difference in the total noises), but not the total number of photons in the two modes. [Contrast this with the mixing operator, eq. (3.1.47).]

Second, the two-mode squeeze operator unitarily transforms the two-mode displacement operator in the following way:

$$S^{\dagger}(r,\varphi)D(\boldsymbol{a},\boldsymbol{\mu})S(r,\varphi) = D(\boldsymbol{a},\boldsymbol{\mu}_{\alpha})\,, \tag{3.1.61}$$

$$\boldsymbol{\mu}_{\alpha} \equiv \boldsymbol{\mu}\cosh r + \sigma_1\boldsymbol{\mu}^{*}\,\mathrm{e}^{2\mathrm{i}\varphi}\sinh r \tag{3.1.62}$$

[eqs. (3.1.29), (3.1.56); cf. eqs. (2.1.26), (2.1.27)]. This shows that the form of the displacement operator is invariant under unitary transformations of $\boldsymbol{a}$ generated by the two-mode squeeze operator:

$$D(\boldsymbol{a},\boldsymbol{\mu}) = D(\boldsymbol{\alpha},\boldsymbol{\mu}_{\alpha})\,. \tag{3.1.63}$$

This equality implies that the TMSS $|\boldsymbol{\mu}_{\alpha}\rangle_{(r,\varphi)}$, defined by eq. (1.19) as $S(r,\varphi)$ acting on the two-mode coherent state $|\boldsymbol{\mu}_{\alpha}\rangle_{\mathrm{coh}}$, can as well be defined as the displacement operator $D(\boldsymbol{a},\boldsymbol{\mu})$ acting on the (two-mode) squeezed vacuum:

$$|\boldsymbol{\mu}_{\alpha}\rangle_{(r,\varphi)} \equiv S(r,\varphi)|\boldsymbol{\mu}_{\alpha}\rangle_{\mathrm{coh}} = D(\boldsymbol{a},\boldsymbol{\mu})S(r,\varphi)|0\rangle\,. \tag{3.1.64}$$

The complex numbers μ_{+}, μ_{-} are the complex amplitudes $\langle a_{+}\rangle$, $\langle a_{-}\rangle$, i.e., $\boldsymbol{\mu}\equiv\langle\boldsymbol{a}\rangle$. They are related to the eigenvalues $\mu_{\alpha+}$, $\mu_{\alpha-}$ by

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\alpha}\cosh r - \sigma_1\boldsymbol{\mu}_{\alpha}^{*}\,\mathrm{e}^{2\mathrm{i}\varphi}\sinh r \tag{3.1.65}$$

[eq. (3.1.56); cf. eqs. (2.1.28)–(2.1.30)]. With these transformations, and the properties described above, one can easily verify the statement made in the Introduction: any state unitarily related to the TMSS $|\boldsymbol{\mu}_{\alpha}\rangle_{(r,\varphi)}$ by a product of rotation and displacement operators is equal to another TMSS (multiplied by an unobservable phase factor) with the same squeeze factor r, but with different squeeze angle and eigenvalues. For example,

$$\begin{aligned}\mathbf{R}(\boldsymbol{\theta})D(\boldsymbol{a},\boldsymbol{\mu}')|\boldsymbol{\mu}_{\alpha}\rangle_{(r,\varphi)} &= \exp[\mathrm{i}\,\mathrm{Im}(\boldsymbol{\mu}'^{\dagger}\boldsymbol{\mu})]D(\boldsymbol{a},\bar{\boldsymbol{\mu}})S(r,\varphi-\theta_{\mathrm{s}})|0\rangle\\ &= \exp[\mathrm{i}\,\mathrm{Im}(\boldsymbol{\mu}'^{\dagger}\boldsymbol{\mu})]|\bar{\boldsymbol{\mu}}_{\alpha}\rangle_{(r,\varphi-\theta_{\mathrm{s}})}\,,\end{aligned}$$

$$\bar{\boldsymbol{\mu}} \equiv \boldsymbol{\mu}(-\boldsymbol{\theta}) + \boldsymbol{\mu}'(-\boldsymbol{\theta}) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}}(\boldsymbol{\mu}+\boldsymbol{\mu}') \tag{3.1.66}$$

[eqs. (3.1.32), (3.1.34), (3.1.60), (3.1.64)].

Finally, it is important to note the formal relation between a product of two single-mode squeeze operators and a two-mode squeeze operator. These unitary operators describe very different physical processes. The product of two single-mode squeeze operators is, roughly speaking, the evolution operator for a process in which two harmonic oscillators are separately squeezed [i.e., each subjected to a degenerate two-photon interaction Hamiltonian (1.8)]. In contrast, a two-mode squeeze operator is the evolution operator for a process in which two harmonic oscillators become correlated through the nondegenerate two-photon interaction Hamiltonian (1.7). Although these operators differ profoundly from each other in a physical sense, they are unitarily equivalent. The unitary operator that transforms them into each other is a mixing operator. In other words, by defining certain linear combinations of a_+ and a_-, call them b_+ and b_-, one can write the two-mode squeeze operator $S(r, \varphi)$ as a product of two single-mode squeeze operators, one for the "b_+ mode" and one for the "b_- mode" [13, 33, 19]. This unitary equivalence is described by the following relation:

$$T(\pm\tfrac{1}{4}\pi, \varphi_a)S_{1+}(r_+, \varphi_+)S_{1-}(r_-, \varphi_-)T^\dagger(\pm\tfrac{1}{4}\pi, \varphi_a) = S_{1+}(r_s, \varphi_+)S_{1-}(r_s, \varphi_-)S(\pm r_a, \varphi_s)\,,$$

$$r_{\substack{s\\a}} \equiv \tfrac{1}{2}(r_+ \pm r_-)\,, \qquad \varphi_{\substack{s\\a}} \equiv \tfrac{1}{2}(\varphi_+ \pm \varphi_-)\,. \tag{3.1.67a}$$

For the special case $r_s \equiv 0$ ($r_+ = -r_-$), this can be rewritten as

$$T(\tfrac{1}{4}\pi, \varphi_a + \tfrac{1}{4}\pi)S_{1+}(r, \varphi_+)S_{1-}(r, \varphi_-)T^\dagger(\tfrac{1}{4}\pi, \varphi_a + \tfrac{1}{4}\pi) = S(r, \varphi_s - \tfrac{1}{4}\pi)\,. \tag{3.1.67b}$$

Thus, one might say there are two ways, mathematically but not physically equivalent, to produce a two-mode squeezed state from two single-mode coherent states. The natural way is simply to turn on a nondegenerate two-photon interaction [eq. (3.1.64)]. The second way, revealed by the relations (3.1.67), is first to turn on a mixing interaction (frequency converter), then separately squeeze the two modes (by turning on two degenerate two-photon interactions), and then turn on another mixing interaction. The relations (3.1.67) thus remind one that the process of separately squeezing two single modes cannot by itself (i.e., without additional frequency-converting processes) produce a state with the reduced noise properties of a two-mode squeezed state. As was pointed out in the Introduction (subsection 1.8; see also subsection 3.1.7), "squeezing" of an electric field – i.e., a reduction in the time-averaged noise in one of its quadrature phases relative to its value for a field composed of modes in coherent states – is possible only if the modes in the field are correlated. That is, squeezing is possible only if each pair of sidebands (models with frequencies $\Omega \pm \varepsilon$) is in a state whose unitary relation to a two-mode coherent state includes a two-mode squeeze operator.

(c) *Two-component vector notation for states with TSQP noise*

The two-mode squeeze operator, $S(r, \varphi)$, and the two rotation operators, $\mathbf{R}(\boldsymbol{\theta})$, represent all the unitary operators that induce matrix transformations on another two-component column vector,

$$\boldsymbol{a}' \equiv \begin{pmatrix} a_+ \\ a_-^\dagger \end{pmatrix}. \tag{3.1.68}$$

The transformation matrices associated with them comprise the (noncompact, pseudounitary) group U(1,1) of all two-dimensional, complex matrices M that preserve the metric σ_3, i.e., that preserve the commutator matrix $[\boldsymbol{a}', \boldsymbol{a}'^\dagger] = \sigma_3$, $M\sigma_3 M^\dagger = \sigma_3$. The elements of this group have the general form

$$M = \exp(i\theta_a)\exp(i\theta_s\sigma_3)C_{r,\varphi} = \exp(i\,\theta_a)C_{r,\varphi+\theta_s}\exp(-i\,\theta_s\sigma_3)$$
$$= \exp(i\theta_a)\begin{pmatrix} \exp(i\theta_s)\cosh r & \exp[i(2\varphi+\theta_s)]\sinh r \\ \exp[-i(2\varphi+\theta_s)]\sinh r & \exp(-i\theta_s)\cosh r \end{pmatrix}, \tag{3.1.69a}$$

for real numbers θ_a, θ_s, r, and φ [cf. eq. (2.3.18d)]. The matrix (3.1.69a) is the transformation matrix that results from a unitary transformation of $\boldsymbol{a}'$ by a product of the two rotation operators and a two-mode squeeze operator, i.e.,

$$M\boldsymbol{a}' = S(r,\varphi)\mathbf{R}(\boldsymbol{\theta})\boldsymbol{a}'\mathbf{R}^\dagger(\boldsymbol{\theta})S^\dagger(r,\varphi) \tag{3.1.69b}$$

[eqs. (3.1.24) and (3.1.55)]. Those transformation matrices associated with unitary transformations on $\boldsymbol{a}'$ by the two-mode squeeze operator $S(r,\varphi)$ and the symmetric rotation operator $R_s(\theta_s)$ form the three-parameter group SU(1, 1), the elements of U(1, 1) with unity determinant. These same transformation matrices describe unitary transformations of the single-mode two-component column vector $\boldsymbol{a}_s$ (see subsection 2.3) by the single-mode rotation and squeeze operators. This similarity is a key to understanding why two-mode squeezed states are the natural analogs of single-mode squeezed states. It is a consequence of the fact that the three operators a_+a_-, $a_-^\dagger a_+^\dagger$, and $(a_+a_+^\dagger + a_-a_-^\dagger)_{\rm sym}$, generators for the two-mode squeeze operator $S(r,\varphi)$ and the symmetric rotation operator $R_s(\theta)$, have the same commutator algebra as the three operators $\frac{1}{2}a^2$, $\frac{1}{2}a^{\dagger 2}$, and $(aa^\dagger)_{\rm sym}$, generators for the single-mode squeeze and rotation operators, $S_1(r,\varphi)$ and $R(\theta)$. Another consequence of this fact is that the mathematical properties of the two-mode squeeze operator and the symmetric rotation operator are identical in form to those of the single-mode squeeze and rotation operators. For example, in addition to the properties mentioned above, one can write factored expressions for the two-mode squeeze operator by analogy with eq. (2.3.15), and the product of two different two-mode squeeze operators as the product of another two-mode squeeze operator and a symmetric rotation operator, by analogy with eq. (2.3.16).

The two-component column vector $\boldsymbol{a}'$ of eq. (3.1.68) provides a basis for a complete description of all states (pure or mixed) with TSQP noise (see the discussion of TSQP noise in subsection 1.8). Another useful basis is the two-component column vector $\boldsymbol{\mathcal{A}}$ formed from the quadrature-phase amplitudes α_1 and α_2:

$$\boldsymbol{\mathcal{A}} \equiv \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = A\lambda\boldsymbol{a}', \tag{3.1.70a}$$

$$A \equiv 2^{-1/2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}, \qquad \lambda \equiv \begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}, \qquad \lambda_\pm \equiv \left(\frac{\Omega\pm\varepsilon}{\Omega}\right)^{1/2} \tag{3.1.70b}$$

[cf. eqs. (1.21)]. The statistics of the quadrature-phase amplitudes provide direct information about the quadrature phases of the electric field. A formalism for describing states with TSQP noise, based on these two-component column vectors, is developed in refs. [18–20]. Note that, by contrast, a formalism capable of describing all two-mode GPS would require a four-component vector notation; such a formalism is described in subsection 3.3 of this paper.

3.1.6. *Product of three squeeze operators*

Consider now the unitary operator $\mathbf{S}$ that relates the most general normalized two-mode GPS to a two-mode coherent state:

$$\mathbf{S} \equiv S_{1+}(r_+, \varphi_+) S_{1-}(r_-, \varphi_-) S(r, \varphi) \tag{3.1.71}$$

[eq. (1.16)]. Note that the inverse of $\mathbf{S}$ is obtained by changing the signs of r_+, r_-, and r and reversing the order of the squeeze operators:

$$\mathbf{S}^{-1} = \mathbf{S}^\dagger = S(-r, \varphi) S_{1-}(-r_-, \varphi_-) S_{1+}(-r_+, \varphi)\,. \tag{3.1.72}$$

The operator $\mathbf{S}$ transforms the operator column vector $\boldsymbol{a}$ into a linear combination of $\boldsymbol{a}$ and $\boldsymbol{a}^*$:

$$\mathbf{S}\boldsymbol{a}\mathbf{S}^\dagger = P_c \boldsymbol{a} + P_s \boldsymbol{a}^* \equiv \boldsymbol{g} \equiv \begin{pmatrix} g_+ \\ g_- \end{pmatrix}; \tag{3.1.73a}$$

the complex matrices P_c and P_s are equal to

$$P_c \equiv \cosh r P_{1c} + e^{2i\varphi} \sinh r \sigma_1 P_{1s}^*\,, \tag{3.1.73b}$$

$$P_s \equiv \cosh r P_{1s} + e^{2i\varphi} \sinh r \sigma_1 P_{1c}^* \tag{3.1.73c}$$

[eqs. (3.1.52) and (3.1.55b); see also eq. (3.2.35) below]. States unitarily related to eigenstates of a_+ and a_- by $\mathbf{S}$ are eigenstates of the transformed annihilation operators g_+ and g_-. The unitarity of $\mathbf{S}$ ensures that the commutator algebra of $g_\pm$ and $g_\pm{}^\dagger$ is identical to that of $a_\pm$ and $a_\pm{}^\dagger$:

$$[\boldsymbol{g}, \boldsymbol{g}^\dagger] = [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \mathbf{1}\,, \tag{3.1.74a}$$

$$[\boldsymbol{g}, \boldsymbol{g}^\mathrm{T}] = [g_+, g_-] i\sigma_2 = [\boldsymbol{a}, \boldsymbol{a}^\mathrm{T}] = 0 \tag{3.1.74b}$$

[eq. (3.1.9a)]. Because of the presence of both single-mode squeeze operators, $\mathbf{S}$ does not, in general, preserve any of the noise moments of $a_\pm$ or $a_\pm{}^\dagger$. The components of the noise matrices T and Q for the general two-mode GPS $|\boldsymbol{\mu}_g\rangle \equiv \mathbf{S}|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}}$ are given below [eqs. (3.1.85)].

Many properties of $\mathbf{S}$ follow trivially from the corresponding properties of the three squeeze operators (e.g., the way it is unitarily transformed by rotation operators); these are left to the reader. Two properties, however, deserve special attention. First, $\mathbf{S}$ unitarily transforms the two-mode displacement operator in the following way:

$$\mathbf{S}^\dagger D(\boldsymbol{a}, \boldsymbol{\mu}) \mathbf{S} = D(\boldsymbol{a}, \boldsymbol{\mu}_g)\,, \tag{3.1.75}$$

$$\boldsymbol{\mu}_g \equiv P_c \boldsymbol{\mu} + P_s \boldsymbol{\mu}^* \tag{3.1.76}$$

[eqs. (3.1.29), (3.1.73)]. This relation reflects the fact the form of the two-mode displacement operator is invariant under unitary transformations of $\boldsymbol{a}$ that are linear in $\boldsymbol{a}$ and $\boldsymbol{a}^*$ (and that do not add to $\boldsymbol{a}$ a constant column vector). Such unitary transformations are generated only by (products of) rotation, mixing, and squeeze operators. The invariance under transformations generated by rotation, mixing, and two-mode squeeze operators has already been noted [eqs. (3.1.33), (3.1.48), (3.1.63)]. The invariance under transformations generated by $\mathbf{S}$ says that

$$D(\boldsymbol{a}, \boldsymbol{\mu}) = D(\boldsymbol{g}, \boldsymbol{\mu}_g)\,. \tag{3.1.77}$$

This equality implies that the general two-mode GPS $|\boldsymbol{\mu}_g\rangle$, defined by eq. (1.18) as the operator $\mathbf{S}$ acting on the two-mode coherent state $|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}}$, can as well be defined as the product of the displacement operator $D(\boldsymbol{a}, \boldsymbol{\mu})$ and $\mathbf{S}$ acting on the vacuum state:

$$|\boldsymbol{\mu}_g\rangle \equiv \mathbf{S}|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}} = D(\boldsymbol{a}, \boldsymbol{\mu})\mathbf{S}|0\rangle . \tag{3.1.78}$$

The complex numbers μ_+, μ_- are the complex amplitudes $\langle a_+\rangle$, $\langle a_-\rangle$, i.e., $\boldsymbol{\mu} \equiv \langle \boldsymbol{a}\rangle$. They are related to the eigenvalues μ_{g+}, μ_{g-} by

$$\boldsymbol{\mu} = P_{\mathrm{c}}^{\dagger}\boldsymbol{\mu}_g - P_{\mathrm{s}}^{\mathrm{T}}\boldsymbol{\mu}_g^{*} . \tag{3.1.79}$$

The second important property to note is that the product of two different operators $\mathbf{S}$ and $\mathbf{S}'$ can always be expressed as another operator $\bar{\mathbf{S}}$ (i.e., another product of the three squeeze operators), multiplied by a mixing and a rotation operator (and an overall phase factor). This property is a consequence of the general fact (proved in subsection 3.3 and appendix A) that any unitary operator whose (Hermitian) generator is composed of bilinear combinations of creation and annihilation operators for two modes can be expressed as a product of three squeeze operators (i.e., an operator like $\mathbf{S}$), a mixing and a rotation operator, and an overall phase factor.

It is proved in the next section, by considering the most general two-mode Gaussian wave function, that the Hermitian generator $H_g^{(2)}$ of the unitary operator $U_g^{(2)} \equiv \exp(-\mathrm{i}H_g^{(2)}t)$ that relates a two-mode GPS to the vacuum state is a sum of linear and bilinear combinations of a_+, a_-, $a_+^{\dagger}$, and $a_-^{\dagger}$. In other words, the most general two-mode GPS is produced when two harmonic oscillators in their ground states are exposed to the interaction Hamiltonians $H_{\mathrm{R}}^{(2)}(t)$, $H_1^{(2)}(t)$, and $H_2^{(2)}(t)$ described in the introduction [eqs. (1.3) and (1.4)]. The unitary operator $U_g^{(2)}$ factors into a product of displacement, squeeze, mixing, and rotation operators (in any order), and an (unobservable) overall phase factor (see subsection 3.3 and appendix A). The properties described in this and the previous sections ensure that any product of rotation, mixing, squeeze and displacement operators can be expressed as the product of a displacement operator and an operator like $\mathbf{S}$ (in either order), multiplied on the right by a rotation and a mixing operator (and an overall phase factor). Since the rotation and mixing operators have no effect on the vacuum state, this shows that the most general two-mode GPS is the state $|\boldsymbol{\mu}_g\rangle$ defined by eqs. (3.1.71) and (3.1.78).

3.1.7. Two-mode GPS and minimum-uncertainty states (MUS)

The noise matrices T and Q (or S_x, S_p, and S_{xp}) for two-mode states that are tensor products of two single-mode states are diagonal matrices. For a product of two single-mode squeezed states, the most general of this type of two-mode GPS, the noise matrices T and Q are

$$T = -\tfrac{1}{2}\begin{pmatrix} \mathrm{e}^{2\mathrm{i}\varphi_+}\sinh 2r_+ & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\varphi_-}\sinh 2r_- \end{pmatrix}, \qquad Q = \tfrac{1}{2}\begin{pmatrix} \cosh 2r_+ & 0 \\ 0 & \cosh 2r_- \end{pmatrix} \tag{3.1.80}$$

[eqs. (2.1.34)].

For a two-mode squeezed state, the noise matrices are not diagonal, but they, too, have simple forms. They follow directly from the transformations (3.1.55) and the noise matrices for a coherent state. For the TMSS $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$ [eq. (3.1.64)] they are

$$T = -\tfrac{1}{2}\,e^{2i\varphi}\sinh 2r\sigma_1\,, \qquad Q = \tfrac{1}{2}\cosh 2r\,\mathbf{1}\,; \tag{3.1.81}$$

$$S_x = \tfrac{1}{2}\begin{pmatrix} \cosh 2r & -\sinh 2r\cos 2\varphi \\ -\sinh 2r\cos 2\varphi & \cosh 2r \end{pmatrix}, \tag{3.1.82a}$$

$$S_p = \sigma_3 S_x \sigma_3 = \tfrac{1}{2}\begin{pmatrix} \cosh 2r & -\sinh 2r\cos 2\varphi \\ -\sinh 2r\cos 2\varphi & \cosh 2r \end{pmatrix}, \tag{3.1.82b}$$

$$S_{xp} = -\tfrac{1}{2}\sinh 2r\sin 2\varphi\,\sigma_1\,. \tag{3.1.82c}$$

The simple forms for these noise matrices are a consequence of TSQP noise, which requires that the noise moments $\langle(\Delta a_+)^2\rangle$, $\langle(\Delta a_-)^2\rangle$, and $\langle\Delta a_+\,\Delta a_-^{\dagger}\rangle$ vanish, or, equivalently, that the noise moments $\langle(\Delta\alpha_1)^2\rangle$, $\langle(\Delta\alpha_2)^2\rangle$, and $\langle\Delta\alpha_2\,\Delta\alpha_2\rangle$ of the quadrature-phase amplitudes vanish [eqs. (1.21) or (3.1.70)]. The nonvanishing second-order noise moments of the quadrature-phase amplitudes for a two-mode squeezed state are

$$\langle|\Delta\alpha_1|^2\rangle = \tfrac{1}{2}[\cosh 2r - (1-\varepsilon^2/\Omega^2)^{1/2}\sinh 2r\cos 2\varphi]\,, \tag{3.1.83a}$$

$$\langle|\Delta\alpha_2|^2\rangle = \tfrac{1}{2}[\cosh 2r + (1-\varepsilon^2/\Omega^2)^{1/2}\sinh 2r\cos 2\varphi]\,, \tag{3.1.83b}$$

$$\langle\Delta\alpha_1\,\Delta\alpha_2^{\dagger}\rangle_{\rm sym} = \tfrac{1}{2}(1-\varepsilon^2/\Omega^2)^{1/2}\sinh 2r\sin 2\varphi + \tfrac{1}{2}i\varepsilon/\Omega\cosh 2r\,, \tag{3.1.83c}$$

[cf. eqs. (2.1.35)]. The squeezing effect is obvious if one rewrites the mean-square uncertainties [eqs. (3.1.83a,b)], for $\varepsilon/\Omega \ll 1$ and $\varphi = 0$, as

$$\langle|\Delta\alpha_{\substack{1\\2}}|^2\rangle = \tfrac{1}{2}[e^{\mp 2r} \pm \tfrac{1}{2}\varepsilon^2/\Omega^2] \cong \tfrac{1}{2}e^{\mp 2r}\,. \tag{3.1.83d}$$

By contrast, two separately squeezed (uncorrelated) single modes do not exhibit this kind of squeezing; the mean-square uncertainties both in quadrature phases are greater than the coherent-state value of $\tfrac{1}{2}$:

$$\langle|\Delta\alpha_{\substack{1\\2}}|^2\rangle = \tfrac{1}{2}(1+\lambda_+^2\sinh^2 r_+ + \lambda_-^2\sinh^2 r_-)\,, \qquad \lambda_\pm \equiv (1+\varepsilon/\Omega)^{1/2}\,. \tag{3.1.84}$$

The noise matrices for the general two-mode GPS $|\boldsymbol{\mu}_g\rangle$ [eq. (3.1.78)] can be obtained from the transformation (3.1.73a) and the noise matrices for a coherent state, or from eqs. (3.2.34) and (3.2.35) below. The components of T and Q, the noise moments of $a_\pm$ and $a_\pm^{\dagger}$, are

$$\begin{aligned}
&\langle(\Delta a_\pm)^2\rangle = -\tfrac{1}{2}\cosh 2r\,e^{2i\varphi_\pm}\sinh 2r_\pm\,,\\
&\langle\Delta a_+\,\Delta a_-\rangle = -\tfrac{1}{2}\sinh 2r(e^{2i\varphi}\cosh r_+\cosh r_- + \exp[2i(\varphi_+ + \varphi_- - \varphi)]\sinh r_+\sinh r_-)\,,\\
&\langle\Delta a_+\,\Delta a_-^{\dagger}\rangle = \tfrac{1}{2}\sinh 2r(\exp[2i(\varphi-\varphi_-)]\cosh r_+\sinh r_- + \exp[-2i(\varphi-\varphi_+)]\cosh r_-\sinh r_+)\,,\\
&\langle|\Delta a_\pm|^2\rangle = \tfrac{1}{2}\cosh 2r\cosh 2r_\pm\,.
\end{aligned} \tag{3.1.85}$$

The choice for the order of the three squeeze operators in $\mathbf{S}$ [eq. (3.1.71)] was made so that the noise moments $\langle(\Delta a_\pm)^2\rangle$ and $\langle|\Delta a_\pm|^2\rangle$ would have these simple forms.

By analogy with single-mode MUS, the natural definition for two-mode "minimum-uncertainty states" (MUS) is those states for which

$$S_x S_p = \tfrac{1}{4}\mathbf{1} \tag{3.1.86}$$

[eq. (3.1.15a); cf. eq. (2.1.36)]. These are (two-mode) GPS that satisfy

$$\operatorname{Im} T = \operatorname{Im} Q = S_{xp} = 0 \tag{3.1.87}$$

[eqs. (3.1.12); cf. eq. (2.1.37)]. The condition (3.1.86) implies that for a two-mode MUS there are only three independent real parameters associated with the ten second-order noise moments. The conditions (3.1.87) are equivalent to the following four conditions on the second-order noise moments of $a_\pm$ and $a_\pm{}^\dagger$:

$$\operatorname{Im}\langle \Delta a_+ \, \Delta a_-{}^\dagger \rangle = 0\,, \tag{3.1.88a}$$

$$\operatorname{Im}\langle \Delta a_+ \, \Delta a_- \rangle = 0\,, \tag{3.1.88b}$$

$$\operatorname{Im}\langle (\Delta a_+)^2 \rangle = 0\,, \tag{3.1.88c}$$

$$\operatorname{Im}\langle (\Delta a_-)^2 \rangle = 0\,. \tag{3.1.88d}$$

Comparison with eqs. (3.1.85) shows that the set of two-mode MUS consists of all two-mode GPS for which $\varphi_+ = \varphi_- = \varphi = 0$ [eq. (3.1.78)]. It is shown below [see eqs. (3.2.5) or (3.2.6)] that a two-mode state is a MUS if and only if it is an eigenstate of both components of the vector linear combination $\hat{\boldsymbol{x}} + \mathrm{i} M_1^{-1} \hat{\boldsymbol{p}}$, where M_1 is a real, symmetric, positive-definite matrix. A two-mode GPS with $\varphi_+ = \varphi_- = \varphi = 0$ satisfies this condition with the matrix M_1 equal to

$$M_1 = \begin{pmatrix} \exp(2r_+)\cosh 2r & \exp(r_+ + r_-)\sinh 2r \\ \exp(r_+ + r_-)\sinh 2r & \exp(2r_-)\cosh 2r \end{pmatrix}. \tag{3.1.89}$$

Milburn [33] has proposed a more restrictive definition of two-mode MUS, which stipulates that the real, symmetric, positive-definite matrix M_1 also be diagonal; his definition therefore includes all products of single-mode squeezed states with $\varphi_+ = \varphi_- = 0$, but it excludes all two-mode states in which the two modes are correlated – i.e., all two-mode GPS for which $r \neq 0$. A two-mode state is a Gaussian pure state if and only if it is an eigenstate of both components of a vector linear combination $\hat{\boldsymbol{x}} + \mathrm{i} M^{-1} \hat{\boldsymbol{p}}$, where M is a two-dimensional complex, symmetric matrix, and $\operatorname{Re}(M)$ is positive definite (see subsection 3.2). All states unitarily related to two-mode MUS by products of rotation and mixing operators satisfy this condition. Conversely, all states that satisfy this condition (i.e., all two-mode GPS) are related to two-mode MUS by products of rotation and mixing operators. Thus, by extending the definition (3.1.86) of two-mode MUS to include all states related to two-mode MUS by products of rotation and mixing operators, one obtains all two-mode GPS. Another way to see this is to note that the four conditions (3.1.88) can always be met for some operators defined as linear combinations of a_+ and a_- by a transformation like (3.1.44b), with appropriate choices for the four parameters q, χ, θ_+, and θ_-.

Another set of two-mode states, the two-mode analogs of single-mode states with random-phase noise, consists of states whose noise moments are invariant under all rotations, i.e., under unitary transformations generated by the rotation operators $\mathbf{R}(\boldsymbol{\theta}) = R_\mathrm{s}(\theta_\mathrm{s}) R_\mathrm{a}(\theta_\mathrm{a})$ [eqs. (3.1.19)]. Two-mode states with random-phase noise are simply tensor products of two single-mode states with random phase noise [see eqs. (3.1.24) and subsection 2.1.5]: all correlated noise moments between the two modes must vanish, and all noise moments of a_+ and a_- separately vanish [eq. (2.1.39a)]. Fields composed of such states have "time-stationary" (TS) noise; the adjective TS is often used, instead of "random phase", to describe the noise associated with these states. For two-mode states (pure or mixed) with Gaussian noise statistics, the conditions for random-phase noise are that the noise matrix $T = 0$, and the noise matrix Q be diagonal [eqs. (3.1.28)]. Note that the condition $T = 0$ is equivalent to the condition

$$S_x(\boldsymbol{\theta}) = S_p(\boldsymbol{\theta}) = S_x(\boldsymbol{\theta}') \quad \text{for all } \theta_+,\ \theta_-,\ \theta'_+,\ \theta'_- , \tag{3.1.90a}$$

where

$$S_x(\boldsymbol{\theta}) \equiv \langle \Delta\hat{\boldsymbol{x}}(\boldsymbol{\theta})\, \Delta\hat{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{\theta})\rangle \equiv \langle \mathbf{R}(\boldsymbol{\theta})\, \Delta\hat{\boldsymbol{x}}\, \Delta\hat{\boldsymbol{x}}^{\mathrm{T}} \mathbf{R}^{\dagger}(\boldsymbol{\theta})\rangle , \tag{3.1.90b}$$

$$S_p(\boldsymbol{\theta}) \equiv \langle \Delta\hat{\boldsymbol{p}}(\boldsymbol{\theta})\, \Delta\hat{\boldsymbol{p}}^{\mathrm{T}}(\boldsymbol{\theta})\rangle \equiv \langle \mathbf{R}(\boldsymbol{\theta})\, \Delta\hat{\boldsymbol{p}}\, \Delta\hat{\boldsymbol{p}}^{\mathrm{T}} \mathbf{R}^{\dagger}(\boldsymbol{\theta})\rangle \tag{3.1.90c}$$

[eqs. (3.1.12), (3.1.24b,c); cf. eq. (2.1.39b)]. This condition alone is not adequate to define random-phase noise, since it does not ensure invariance of all second-order noise moments under rotations induced by the antisymmetric rotation operator $R_{\mathrm{a}}(\theta)$; that invariance requires the additional condition that the noise matrix Q be diagonal [eqs. (3.1.28)]. The only nonvanishing second-order noise moments for a two-mode state with random-phase noise are the total noises for each mode, $\langle|\Delta a_+|^2\rangle$ and $\langle|\Delta a_-|^2\rangle$. The intersection between the two-mode Gaussian pure states and states with random-phase noise is the set of two-mode coherent states, which have

$$T_{\mathrm{coh}} = (S_{xp})_{\mathrm{coh}} = 0 , \tag{3.1.91a}$$

$$Q_{\mathrm{coh}} = (S_x)_{\mathrm{coh}} = (S_p)_{\mathrm{coh}} = \tfrac{1}{2}\mathbf{1} \tag{3.1.91b}$$

[cf. eqs. (2.1.33)].

Another important set of two-mode states consists of those with TSQP (time-stationary quadrature-phase) noise (see subsection 1.8). These are states whose noise moments are invariant under rotations induced by the antisymmetric rotation operator $R_{\mathrm{a}}(\theta)$ [eq. (3.1.18b)] [50]. The noise moments of $a_\pm$ and $a_\pm{}^\dagger$ for two-mode states with TSQP noise satisfy the conditions

$$\langle(\Delta a_\pm)^r(\Delta a_\pm{}^\dagger)^s\rangle_{\mathrm{sym}} = 0 \quad \text{if } r \neq s , \tag{3.1.92a}$$

$$\langle(\Delta a_+)^r(\Delta a_-)^s\rangle = 0 \quad \text{if } r \neq s , \tag{3.1.92b}$$

$$\langle(\Delta a_+)^r(\Delta a_-{}^\dagger)^s\rangle = 0 , \tag{3.1.92c}$$

where r and s are nonnegative integers. For the quadrature-phase amplitudes, these conditions are

$$\langle(\Delta\alpha_1)^r(\Delta\alpha_2)^s\rangle = 0 , \tag{3.1.93a}$$

$$\langle(\Delta\alpha_m)^r(\Delta\alpha_n{}^\dagger)^s\rangle_{\mathrm{sym}} = 0 \quad \text{if } r \neq s , \tag{3.1.93b}$$

where $m, n = 1, 2$. When these conditions are satisfied, all time-dependent noise moments of the electric-field quadrature phases E_1 and E_2 vanish – hence the adjective TSQP. Comparison of eqs. (3.1.93) with eq. (2.1.39a) suggests that the adjective TSQP, which describes the noise in a field composed of these states, could be replaced by "random-quadrature-phase" when describing the noise associated with these two-mode states. For states (pure or mixed) with Gaussian noise statistics, the conditions (3.1.92) for TQSP noise reduce to

$$\langle \Delta a_+ \, \Delta a_-^\dagger \rangle = \langle (\Delta a_+)^2 \rangle = \langle (\Delta a_-)^2 \rangle = 0 \; ; \tag{3.1.94a}$$

i.e., the noise matrix T has only off-diagonal components, and the noise matrix Q is diagonal [eqs. (3.1.28)]. For the quadrature-phase amplitudes, the conditions for Gaussian TSQP noise are

$$\langle (\Delta \alpha_1)^2 \rangle = \langle (\Delta \alpha_2)^2 \rangle = \langle \Delta \alpha_1 \, \Delta \alpha_2 \rangle = 0 \, . \tag{3.1.94b}$$

The intersection between two-mode GPS and states with TSQP noise is the set of two-mode squeezed states.

Note that TSQP noise is less restrictive than TS noise – i.e., time-stationary noise in the quadrature phases E_1 and E_2 does not imply time-stationary noise in the total electric field E. Like TS noise, TSQP noise requires that each mode have random-phase noise [$\langle (\Delta a_\pm)^2 \rangle = 0$; eq. (3.1.94a)]; but, unlike TS noise, it does not require that the modes be uncorrelated. It is precisely the nonvanishing correlated noise moment $\langle \Delta a_+ \, \Delta a_- \rangle$ that makes possible the squeezing effect.

3.2. Two-mode Gaussian wave functions

This section begins with the most general two-mode Gaussian coordinate-space wave function, with all parameters arbitrary, subject to normalization. Subsection 3.2.1 explores the relation of the parameters in the wave function to the noise properties of a two-mode GPS, and derives general relations between the different noise moments. The two-component vector notation defined in the preceding section (subsection 3.1.1b,c) allows these relations to be derived and expressed very simply as matrix relations between the different second-order noise matrices. Subsection 3.2.2 examines the pairs of independent operators of which two-mode GPS are eigenstates, these being determined by the wave function. It establishes thereby the logical connection between the wave function and the formal definition of a two-mode GPS as a unitary operator acting on the (two-mode) vacuum state. Subsection 3.2.3 derives the unitary operator that relates the most general two-mode GPS to the vacuum state, and shows that it is equal to a product of two single-mode squeeze operators and one two-mode squeeze operator. The parameters in the general two-mode Gaussian wave function are given explicitly in terms of the six real parameters associated with these squeeze operators. Subsection 3.2.4 considers the most general two-mode Gaussian momentum-space wave function, and relates its parameters to those of the coordinate-space wave function.

3.2.1. The wave function

Consider now the coordinate-space wave function for the most general two-mode Gaussian pure state, symbolized here by the state vector $|\boldsymbol{\mu}_g\rangle$ (or $|\mu_{g+}, \mu_{g-}\rangle$). The two-mode GPS $|\boldsymbol{\mu}_g\rangle$ is an eigenstate of a pair of operators g_+ and g_-, whose general forms are discussed below, with complex eigenvalues μ_{g+} and μ_{g-}, respectively. [It will be seen below that g_+ and g_- have the forms discussed in subsection 3.1.6.] The wave function is written in terms of the dimensionless position variables $x_\pm$, the eigenvalues of the Hermitian operators $\hat{x}_\pm$. The most general (normalized) two-mode Gaussian coordinate-space wave function has the form

$$\begin{aligned}\langle x_+x_-|\boldsymbol{\mu}_g\rangle &= N_g \exp(\tfrac{1}{2}\mathrm{i}\delta_x)\exp[-\tfrac{1}{2}\mathrm{i}(p_{0+}x_{0+}+p_{0-}x_{0-})]\exp[\mathrm{i}(p_{0+}x_+ + p_{0-}x_-)]\\ &\quad\times\exp[-\tfrac{1}{2}M_{11}(\Delta x_+)^2-\tfrac{1}{2}M_{22}(\Delta x_-)^2-\tfrac{1}{2}(M_{12}+M_{21})\,\Delta x_+\,\Delta x_-]\\ &= N_g\exp(\tfrac{1}{2}\mathrm{i}\delta_x)\exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}_0{}^{\mathrm{T}}\boldsymbol{x}_0)\exp(\mathrm{i}\boldsymbol{p}_0{}^{\mathrm{T}}\boldsymbol{x})\exp(-\tfrac{1}{2}\,\Delta\boldsymbol{x}^{\mathrm{T}}M\,\Delta\boldsymbol{x})\,,\end{aligned}$$

$$\Delta\boldsymbol{x}\equiv\begin{pmatrix}\Delta x_+\\ \Delta x_-\end{pmatrix},\qquad \Delta x_\pm \equiv x_\pm - \langle\hat{x}_\pm\rangle \tag{3.2.1}$$

[cf. eq. (2.2.1)]. Here $\boldsymbol{x}_0$ and $\boldsymbol{p}_0$ are the column vectors for the mean position and momentum [eqs. (3.1.7)], with components defined by

$$x_{0\pm}\equiv\langle\hat{x}_\pm\rangle=\int_{-\infty}^{\infty}\mathrm{d}x_+\int_{-\infty}^{\infty}\mathrm{d}x_-\;x_\pm|\langle x_+x_-|\boldsymbol{\mu}_g\rangle|^2\,, \tag{3.2.2a}$$

$$p_{0\pm}\equiv\langle\hat{p}_\pm\rangle=-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x_+\int_{-\infty}^{\infty}\mathrm{d}x_-\langle\boldsymbol{\mu}_g|x_+x_-\rangle\partial_{x_\pm}\langle x_+x_-|\boldsymbol{\mu}_g\rangle\,, \tag{3.2.2b}$$

M is a two-dimensional, complex matrix whose components M_{ij} are related to the second-order noise moments of $\hat{x}_\pm$ and $\hat{p}_\pm$, δ_x is an unobservable phase angle (separated out for reasons discussed below), and N_g is a (real) normalization constant determined by the condition $\langle\boldsymbol{\mu}_g|\boldsymbol{\mu}_g\rangle=1$. The subscript "$x$" on the phase angle δ_x serves only to distinguish δ_x from the phase angle δ_p which appears in the momentum-space wave function [eqs. (3.2.38)–(3.2.41) below]; δ_x has no dependence on $x_\pm$. The antisymmetric part of the matrix M is irrelevant, so I assume henceforth that M is symmetric, $M\equiv M^{\mathrm{T}}$ ($M_{12}\equiv M_{21}$). Normalizability dictates that the real part of M be positive definite – i.e., that

$$\operatorname{Tr}M_1>0\,,\qquad \det M_1>0\,,\qquad M_1\equiv\operatorname{Re}M=\tfrac{1}{2}(M+M^*)\,, \tag{3.2.3}$$

and the normalization constant N_g is equal to

$$N_g=(\pi^2/\det M_1)^{-1/4} \tag{3.2.4}$$

[cf. eqs. (2.2.2)–(2.2.4)].

The most important parameters in the wave function (3.2.1) are the three complex numbers that make up the (symmetric) matrix M. The form of the wave function tells one that the state $|\boldsymbol{\mu}_g\rangle$ is a simultaneous eigenstate of both components of the vector linear combination $\hat{\boldsymbol{x}}+\mathrm{i}M^{-1}\hat{\boldsymbol{p}}$, and hence that the matrix M is related to the noise matrices S_x, S_p, and S_{xp} by

$$M\equiv M_1+\mathrm{i}M_2=-\mathrm{i}S_x{}^{-1}(S_{xp}+\tfrac{1}{2}\mathrm{i}\mathbf{1})=-\mathrm{i}S_p(S_{xp}-\tfrac{1}{2}\mathrm{i}\mathbf{1})^{-1}\,. \tag{3.2.5a}$$

The real and imaginary parts of M are therefore equal to

$$M_1 \equiv \mathrm{Re}\, M = \tfrac{1}{2} S_x^{-1}\,, \qquad M_2 \equiv \mathrm{Im}\, M = -S_x^{-1} S_{xp}\,, \tag{3.2.5b}$$

and the absolute square of its determinant is

$$|\det M|^2 = \det S_p / \det S_x \tag{3.2.5c}$$

[cf. eqs. (2.2.5)]. Equation (3.2.5c) shows that for normalizable two-mode GPS the matrix M must be nonsingular [if $\det M = 0$, the wave function in the momentum representation is a delta function; see discussion below eqs. (3.1.11)]. Inverting these expressions gives the covariance matrices S_x, S_p, and S_{xp} in terms of the matrix M:

$$S_x = (2M_1)^{-1}\,, \qquad S_p = [2\,\mathrm{Re}(M^{-1})]^{-1}\,, \qquad S_{xp} = -(2M_1)^{-1} M_2 \tag{3.2.6}$$

[cf. eq. (2.2.6)]. The normalization constant N_g can thus be rewritten as

$$N_g = (\pi^2/\det M_1)^{-1/4} = (4\pi^2 \det S_x)^{-1/4} \tag{3.2.7}$$

[cf. eq. (2.2.7)].

That the state $|\boldsymbol{\mu}_g\rangle$ is an eigenstate of the components of $\hat{\boldsymbol{x}} + \mathrm{i}M^{-1}\hat{\boldsymbol{p}}$ means that it is also an eigenstate of the components of $\boldsymbol{a} + (M+1)^{-1}(M-1)\boldsymbol{a}^*$. The noise matrices T and Q are therefore more naturally expressed in terms of the symmetric complex matrix

$$\Gamma = \Gamma^{\mathrm{T}} \equiv (M+\mathbf{1})^{-1}(M-1) = -(Q+\tfrac{1}{2}\mathbf{1})^{-1} T = -(Q-\tfrac{1}{2}\mathbf{1}) T^{*-1} \tag{3.2.8}$$

[cf. eq. (2.2.8)]. Inverting these expressions gives the noise matrices T and Q in terms of the matrices Γ and M:

$$\begin{aligned} T &= -\Gamma(\mathbf{1} - \Gamma^*\Gamma)^{-1} = -(\mathbf{1} - \Gamma\Gamma^*)^{-1}\Gamma \\ &= -(M^* + \mathbf{1})(4M_1)^{-1}(M - \mathbf{1})\,, \end{aligned} \tag{3.2.9a}$$

$$Q = \tfrac{1}{2}(\mathbf{1} + \Gamma\Gamma^*)(\mathbf{1} - \Gamma\Gamma^*)^{-1} = (M+\mathbf{1})^{-1}(\mathbf{1} + MM^*)(4M_1)^{-1}(M+\mathbf{1}) \tag{3.2.9b}$$

[cf. eqs. (2.2.9)]. Note also that

$$\mathbf{1} - \Gamma\Gamma^* = (M+\mathbf{1})^{-1} 4M_1 (M^* + \mathbf{1})^{-1}\,; \tag{3.2.10}$$

hence normalizability requires that the (Hermitian) matrix $(\mathbf{1} - \Gamma\Gamma^*)$ be positive definite. The above expressions show, though not in a transparent way, that only six of the ten real pieces of information in the second-order noise moments for a two-mode GPS are independent, since

$$S_x S_p = \tfrac{1}{4}\mathbf{1} + S_{xp}^{\;2}\,, \tag{3.2.11a}$$

$$Q^2 = \tfrac{1}{4}\mathbf{1} + TT^* \tag{3.2.11b}$$

[cf. eqs. (3.1.15) and (2.2.11)]. They also show that the following matrix products are symmetric:

$$S_{xp}S_x = (S_{xp}S_x)^{\mathrm{T}}\,, \qquad S_pS_{xp} = (S_pS_{xp})^{\mathrm{T}}\,, \qquad QT = (QT)^{\mathrm{T}} = TQ^*\,. \tag{3.2.11c}$$

These relations are made more obvious below [eqs. (3.2.30)–(3.2.34)].

The remaining parameter in the wave function (3.2.1) is the phase angle δ_x; in general it can be any real number. The phase angle δ_x is unobservable, but for a state defined as a particular unitary operator acting on the (two-mode) vacuum state it has a well-defined value, provided one assigns a phase angle to the vacuum-state wave functions. The properties of the displacement operator reveal why the phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ separates naturally from the overall phase factor in the wave function $\langle \boldsymbol{x}|\boldsymbol{\mu}_g\rangle$. The definition (3.1.29) of the two-mode displacement operator implies that the wave function for a "displaced" two-mode state $D(\boldsymbol{a}, \boldsymbol{\mu})|\Psi\rangle$ is related to the wave function of the original state $|\Psi\rangle$ in the following way:

$$\langle x|D(\boldsymbol{a}, \boldsymbol{\mu})|\Psi\rangle = \exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}_0{}^{\mathrm{T}}\boldsymbol{x}_0)\exp(\mathrm{i}\boldsymbol{p}_0{}^{\mathrm{T}}\boldsymbol{x})\langle \boldsymbol{x}-\boldsymbol{x}_0|\Psi\rangle \tag{3.2.12}$$

[cf. eq. (2.2.12)]. A natural way, therefore, to obtain an arbitrary two-mode pure state $|\Psi_{\boldsymbol{\mu}}\rangle$ with complex amplitudes μ_+ and μ_- is to operate with the two-mode displacement operator $D(\boldsymbol{a}, \boldsymbol{\mu})$ on a state $|\Psi_o\rangle \equiv \mathbf{U}_0|0\rangle$ that has the desired noise properties but has zero complex amplitudes ($\langle 0|\mathbf{U}_0{}^\dagger \boldsymbol{a}\mathbf{U}_0|0\rangle = 0$):

$$|\Psi_{\boldsymbol{\mu}}\rangle \equiv D(\boldsymbol{a}, \boldsymbol{\mu})\mathbf{U}_0|0\rangle\,. \tag{3.2.13}$$

The property (3.1.31) of the displacement operator then ensures that $|\Psi_{\boldsymbol{\mu}}\rangle$ has complex amplitudes μ_+ and μ_-,

$$\langle \Psi_{\boldsymbol{\mu}}|\boldsymbol{a}|\Psi_{\boldsymbol{\mu}}\rangle = \boldsymbol{\mu}\,. \tag{3.2.14}$$

Any normalized two-mode pure state with complex amplitudes μ_+ and μ_- can be defined by an expression like (3.2.13). The advantage of this definition is that the state's mean values $x_{0\pm}$ and $p_{0\pm}$ (or complex amplitudes $\mu_\pm$) are determined solely by the displacement operator $D(\boldsymbol{a}, \boldsymbol{\mu})$, and its noise moments of $a_\pm$ and $a_\pm{}^\dagger$ are determined solely by the unitary operator $\mathbf{U}_0$. The (normalized) two-mode GPS $|\boldsymbol{\mu}_g\rangle$ with complex amplitudes μ_+, μ_- can therefore be formally defined by

$$|\boldsymbol{\mu}_g\rangle \equiv D(\boldsymbol{a}, \boldsymbol{\mu})\mathbf{U}_g|0\rangle\,. \tag{3.2.15}$$

Note the following three properties of $\mathbf{U}_g$: First, it is uniquely defined only up to (right-hand) multiplication by rotation operators $R_\pm(\theta_\pm)$, a mixing operator $T(q, \chi)$, and an overall phase factor. Second, since it defines the noise moments of $a_\pm$ and $a_\pm{}^\dagger$ (or $\hat{x}_\pm$ and $\hat{p}_\pm$) for the GPS $|\boldsymbol{\mu}_g\rangle$, it has associated with it no more than six independent real parameters (over and above those of the rotation and mixing operators and phase factor). Third, since the state $|\boldsymbol{\mu}_g\rangle$ has complex amplitudes μ_+, μ_-, the expectation value $\langle 0|\mathbf{U}_g{}^\dagger \boldsymbol{a}\mathbf{U}_g|0\rangle$ must vanish.

The phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ in the wave function $\langle \boldsymbol{x}|\Psi_{\boldsymbol{\mu}}\rangle$ is given, from eqs. (3.2.1) and (3.2.12), by

$$\exp(\tfrac{1}{2}\mathrm{i}\delta_x) = \langle x_+ = x_- = 0|\mathbf{U}_g|0\rangle / |\langle x_+ = x_- = 0|\mathbf{U}_g|0\rangle| \tag{3.2.16}$$

[cf. eq. (2.2.16)]. The phase angle δ_x has no dependence on the complex amplitudes $\mu_\pm$, provided $\mathbf{U}_g$ does not; any dependence of $\mathbf{U}_g$ on $\mu_\pm$ is artificial, since it does not affect the state's complex amplitudes $\langle a_\pm \rangle$. Consider, for illustration, the two-mode coherent state $|\boldsymbol{\mu}\rangle_{\rm coh} \equiv D(\boldsymbol{a}, \boldsymbol{\mu})|0\rangle$ [eq. (1.15)], for which the operator $\mathbf{U}_g$ is the identity operator. Equation (3.2.12) says that the wave function for $|\boldsymbol{\mu}\rangle_{\rm coh}$ is related to the vacuum-state wave function $\langle \boldsymbol{x}|0\rangle$ by

$$\langle \boldsymbol{x}|\boldsymbol{\mu}\rangle_{\rm coh} = \exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{p}_0{}^{\rm T}\boldsymbol{x}_0)\exp(\mathrm{i}\boldsymbol{p}_0{}^{\rm T}\boldsymbol{x})\langle \boldsymbol{x}-\boldsymbol{x}_0|0\rangle\,, \tag{3.2.17}$$

so the phase angle δ_x for a two-mode coherent-state wave function is equal to the phase angle assigned to the two-mode vacuum-state wave function.

3.2.2. Operators of which two-mode GPS are eigenstates

The form of its wave function shows that a two-mode GPS $|\boldsymbol{\mu}_g\rangle$ is a simultaneous eigenstate of two linearly independent operators g_+ and g_- that are linear combinations of the components of the vector linear combinations $\hat{\boldsymbol{x}} + \mathrm{i}M^{-1}\hat{\boldsymbol{p}}$ or $\boldsymbol{a} + \Gamma\boldsymbol{a}^*$. The label $\boldsymbol{\mu}_g$ for the GPS $|\boldsymbol{\mu}_g\rangle$ is chosen to be the vector whose components are the eigenvalues of g_+ and g_-. Thus, one can write the following relations:

$$\boldsymbol{g}|\boldsymbol{\mu}_g\rangle = \boldsymbol{\mu}_g|\boldsymbol{\mu}_g\rangle\,, \tag{3.2.18a}$$

$$\boldsymbol{g} = K(\hat{\boldsymbol{x}} + \mathrm{i}M^{-1}\hat{\boldsymbol{p}}) = \bar{K}(\boldsymbol{a} + \Gamma\boldsymbol{a}^*)\,, \tag{3.2.18b}$$

$$\boldsymbol{\mu}_g = K(\boldsymbol{x}_0 + \mathrm{i}M^{-1}\boldsymbol{p}_0) = \bar{K}(\boldsymbol{\mu} + \Gamma\boldsymbol{\mu}^*)\,, \tag{3.2.18c}$$

where K and $\bar{K}$ are two-dimensional nonsingular matrices. It is instructive to consider the general form for all independent operators g_+, g_- of which the two-mode GPS $|\boldsymbol{\mu}_g\rangle$ is an eigenstate:

$$\begin{aligned}\boldsymbol{g} \equiv \begin{pmatrix} g_+ \\ g_- \end{pmatrix} &= P_{\rm c}\boldsymbol{a} + P_{\rm s}\boldsymbol{a}^* \equiv P_{\rm c}(\boldsymbol{a} + \Gamma\boldsymbol{a}^*) \\ &= P_p\hat{\boldsymbol{x}} + \mathrm{i}P_x\hat{\boldsymbol{p}} \equiv P_p(\hat{\boldsymbol{x}} + \mathrm{i}M^{-1}\hat{\boldsymbol{p}})\,.\end{aligned} \tag{3.2.19a}$$

Here P_p, P_x, $P_{\rm c}$, and $P_{\rm s}$ are two-dimensional complex matrices, related to each other by

$$P_{\substack{\rm c\\ \rm s}} = 2^{-1/2}(P_p \pm P_x)\,, \qquad P_{\substack{p\\ x}} = 2^{-1/2}(P_{\rm c} \pm P_{\rm s})\,. \tag{3.2.19b}$$

The eigenvalues μ_{g+}, μ_{g-} are related to the complex amplitudes μ_+, μ_- and the mean positions and momentums $x_{0\pm}$, $p_{0\pm}$ by similar relations,

$$\boldsymbol{\mu}_g = P_p\boldsymbol{x}_0 + \mathrm{i}P_x\boldsymbol{p}_0 = P_{\rm c}\boldsymbol{\mu} + P_{\rm s}\boldsymbol{\mu}^*\,. \tag{3.2.20}$$

Inverting eqs. (3.2.19) and (3.2.20) leads to the following expressions for $\boldsymbol{a}$ and $\boldsymbol{\mu}$ in terms of $\boldsymbol{g}$ and $\boldsymbol{\mu}_g$:

$$\boldsymbol{a} = P_{\rm c}^\dagger Y_g^{-1}\boldsymbol{g} - P_{\rm s}^{\rm T}(Y_g^*)^{-1}\boldsymbol{g}^*\,, \tag{3.2.21a}$$

$$\boldsymbol{\mu} \equiv \langle \boldsymbol{a} \rangle = P_{\mathrm{c}}^{\dagger} Y_g^{-1} \boldsymbol{\mu}_g - P_{\mathrm{s}}^{\mathrm{T}} (Y_g^*)^{-1} \boldsymbol{\mu}_g^* , \tag{3.2.21b}$$

where $Y_g \equiv [\boldsymbol{g}, \boldsymbol{g}^{\dagger}]$ is the Hermitian commutator matrix

$$Y_g \equiv [\boldsymbol{g}, \boldsymbol{g}^{\dagger}] \equiv \boldsymbol{g}\boldsymbol{g}^{\dagger} - (\boldsymbol{g}^*\boldsymbol{g}^{\mathrm{T}})^{\mathrm{T}} = \begin{pmatrix} [g_+, g_+{}^{\dagger}] & [g_+, g_-{}^{\dagger}] \\ [g_-, g_+{}^{\dagger}] & [g_-, g_-{}^{\dagger}] \end{pmatrix} \tag{3.2.21c}$$

[cf. eqs. (2.2.21)]. These relations imply the important equality

$$D(\boldsymbol{a}, \boldsymbol{\mu}) = D(\boldsymbol{g}, Y_g^{-1} \boldsymbol{\mu}_g) \tag{3.2.22}$$

[cf. eq. (2.2.22)].

The equality (3.2.22) enables one to see explicitly how the form of the unitary operator $\mathbf{U}_g$, which defines a two-mode GPS through eq. (3.2.15), is determined by the forms of the operators g_+ and g_-. To see this, begin with an alternative definition for the GPS $|\boldsymbol{\mu}_g\rangle$. First, assume that $|\boldsymbol{\mu}_g\rangle$ is related to the vacuum state by some unitary operator $\bar{\mathbf{U}}$:

$$|\boldsymbol{\mu}_g\rangle = \bar{\mathbf{U}}|0\rangle . \tag{3.2.23a}$$

It is then convenient to define another unitary operator $\mathbf{U}_g$ by

$$\bar{\mathbf{U}} \equiv \mathbf{U}_g D(\boldsymbol{a}, \boldsymbol{\mu}_g) , \tag{3.2.23b}$$

so that the state $|\boldsymbol{\mu}_g\rangle$ is equal to the operator $\mathbf{U}_g$ acting on the two-mode coherent state $|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}}$,

$$|\boldsymbol{\mu}_g\rangle = \mathbf{U}_g D(\boldsymbol{a}, \boldsymbol{\mu}_g)|0\rangle = \mathbf{U}_g|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}} . \tag{3.2.24}$$

[The equality (3.2.22) will be seen to ensure that the operator $\mathbf{U}_g$ defined here is the same as that in eq. (3.2.15).] It is then consistent with the eigenvalue equations (3.2.18a) that the operators g_+ and g_- be unitarily related to the annihilation operators a_+ and a_-, respectively, through the operator $\mathbf{U}_g$:

$$\boldsymbol{g} = \mathbf{U}_g \boldsymbol{a} \mathbf{U}_g{}^{\dagger} . \tag{3.2.25}$$

The form of $\mathbf{U}_g$ is thus determined by the forms of the operators g_+ and g_-. Note that this relation says that both a_+ and a_- are transformed by the same unitary operator, $\mathbf{U}_g$. The fact that a_+ and a_- are transformed by the same, unitary operator ensures that $[\boldsymbol{g}, \boldsymbol{g}^{\dagger}] = [\boldsymbol{a}, \boldsymbol{a}^{\dagger}] = \mathbf{1}$. This, together with the forms (3.2.19a) of g_+ and g_-, implies the equality

$$D(\boldsymbol{a}, \boldsymbol{\mu}) = D(\boldsymbol{g}, \boldsymbol{\mu}_g) \tag{3.2.26}$$

[eq. (3.2.22)], which in turn proves the equivalence of the definitions (3.2.15) and (3.2.24) for $|\boldsymbol{\mu}_g\rangle$. Thus, any two-mode GPS $|\boldsymbol{\mu}_g\rangle$ has the following two equivalent definitions:

$$|\boldsymbol{\mu}_g\rangle = \mathbf{U} D(\boldsymbol{a}, \boldsymbol{\mu}_g)|0\rangle = \mathbf{U}|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}} = D(\boldsymbol{a}, \boldsymbol{\mu})\mathbf{U}|0\rangle \tag{3.2.27}$$

[cf. eq. (2.2.27)].

Return now to the general forms (3.2.19a) for the operators g_+ and g_- of which two-mode GPS are eigenstates. Six of the sixteen real parameters in the expressions (3.2.19a) for $\boldsymbol{g}$ are determined by the wave function $\langle \boldsymbol{x} | \boldsymbol{\mu}_g \rangle$, which specifies the (symmetric) matrix products

$$P_x^{-1} P_p \equiv M\,, \qquad P_c^{-1} P_s \equiv \Gamma\,. \tag{3.2.28}$$

(It will be seen that these parameters specify the squeeze factors r, $r_\pm$ and the squeeze angles φ, $\varphi_\pm$.) Six more real parameters are partially determined by the requirement that g_+ and g_- have a complete (or overcomplete) set of simultaneous, normalizable eigenstates, i.e., (i) that the commutator $[g_+, g_-] = 0$ (or, equivalently, that the antisymmetric commutator matrix $[\boldsymbol{g}, \boldsymbol{g}^{\mathrm{T}}]$ vanish), and (ii) that the Hermitian commutator matrix $[\boldsymbol{g}, \boldsymbol{g}^\dagger]$ be positive definite (for further discussion of this requirement see appendix C). These parameters are determined completely if one specifies that the operators g_+ and g_- be unitarily related to a_+ and a_-, respectively, by the same unitary operator [eq. (3.2.25)], which implies that

$$[\boldsymbol{g}, \boldsymbol{g}^{\mathrm{T}}] = [\boldsymbol{a}, \boldsymbol{a}^{\mathrm{T}}] = 0\,, \tag{3.2.29a}$$

$$[\boldsymbol{g}, \boldsymbol{g}^\dagger] = [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \mathbf{1}\,. \tag{3.2.29b}$$

The antisymmetric commutator matrix $[\boldsymbol{g}, \boldsymbol{g}^{\mathrm{T}}]$ is related to the matrices in the expression (3.2.24a) for $\boldsymbol{g}$ in the following way:

$$\begin{aligned} [\boldsymbol{g}, \boldsymbol{g}^{\mathrm{T}}] &\equiv \boldsymbol{g}\boldsymbol{g}^{\mathrm{T}} - (\boldsymbol{g}\boldsymbol{g}^{\mathrm{T}})^{\mathrm{T}} = [g_+, g_-]\mathrm{i}\sigma_2 \\ &= P_x P_p^{\mathrm{T}} - P_p P_x^{\mathrm{T}} = P_p[M^{-1} - (M^{-1})^{\mathrm{T}}] P_p^{\mathrm{T}} \\ &= P_c P_s^{\mathrm{T}} - P_s P_c^{\mathrm{T}} = P_c(\Gamma^{\mathrm{T}} - \Gamma) P_c^{\mathrm{T}}\,. \end{aligned} \tag{3.2.30}$$

Thus, the condition (3.2.29a) – i.e., that the operators g_+ and g_- commute – simply tells one that the matrix M must be symmetric. The Hermitian commutator matrix $[\boldsymbol{g}, \boldsymbol{g}^\dagger]$ can be written in the following different ways:

$$\begin{aligned} [\boldsymbol{g}, \boldsymbol{g}^\dagger] &= P_x P_p^\dagger + P_p P_x^\dagger = 2 P_x M_1 P_x^\dagger = P_x S_x^{-1} P_x^\dagger \\ &= 2 P_p \operatorname{Re}(M^{-1}) P_p^\dagger = P_p S_p^{-1} P_p^\dagger \\ &= P_c P_c^\dagger - P_s P_s^\dagger = P_c(\mathbf{1} - \Gamma\Gamma^*) P_c^\dagger = P_c(Q + \tfrac{1}{2}\mathbf{1})^{-1} P_c^\dagger \\ &= P_s[(\Gamma^*\Gamma)^{-1} - \mathbf{1}] P_s^\dagger = P_s(Q^* - \tfrac{1}{2}\mathbf{1})^{-1} P_s^\dagger \\ &= P_s \Gamma^{-1}(\mathbf{1} - \Gamma\Gamma^*) P_c^\dagger = -P_s T^{-1} P_c^\dagger \end{aligned} \tag{3.2.31}$$

[eqs. (3.2.5) and (3.2.8); cf. eq. (2.2.30)]. Thus, the condition that g_+ and g_- be unitarily related to a_+ and a_- through the same unitary operator [eq. (3.2.29b)] implies that

$$P_x P_p^\dagger + P_p P_x^\dagger = P_c P_c^\dagger - P_s P_s^\dagger = \mathbf{1}\,. \tag{3.2.32}$$

The expressions (3.2.30) and (3.2.31) show that the operators g_+ and g_- have simultaneous normalizable eigenstates – i.e., $[g_+, g_-] = 0$ and the Hermitian commutator matrix $[\boldsymbol{g}, \boldsymbol{g}^\dagger]$ is positive-definite – if and only if the real matrices M_1 and $\mathbf{1} - \Gamma\Gamma^*$ are symmetric and positive-definite; this is equivalent to the condition that the wave function $\langle \boldsymbol{x} \mid \boldsymbol{\mu}_g\rangle$ be normalizable [eqs. (3.2.3) and (3.2.10)]. This condition also requires that the matrices P_p, P_x, and P_s be nonsingular. Two other properties of the matrices P_p, P_x, P_c, and P_s can be found from the expression (3.2.21a) for $\boldsymbol{a}$ in terms of $\boldsymbol{g}$ and $\boldsymbol{g}^*$, by setting $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \mathbf{1}$. For the matrices P_p and P_x these properties are

$$\mathrm{Im}(P_p{}^\dagger Y_g^{-1} P_p) = \mathrm{Im}(P_x{}^\dagger Y_g^{-1} P_x) = 0\,, \tag{3.2.33a}$$

$$\mathrm{Re}(P_x{}^\dagger Y_g^{-1} P_p) = \tfrac{1}{2}\mathbf{1}\,. \tag{3.2.33b}$$

For the matrices P_c and P_s they are

$$P_c{}^\dagger Y_g^{-1} P_s = (P_c{}^\dagger Y_g^{-1} P_s)^{\mathrm{T}}\,, \tag{3.2.33c}$$

$$P_c{}^\dagger Y_g^{-1} P_c - P_s{}^{\mathrm{T}} (Y_g^{-1})^* P_s{}^* = \mathbf{1}\,. \tag{3.2.33d}$$

From now on I restrict attention to operators g_+, g_- that are unitarily related to a_+ and a_- by the same unitary transformation [eq. (3.2.25)], so that $[\boldsymbol{g}, \boldsymbol{g}^\dagger] = \mathbf{1}$. This entails no loss of generality, since by taking appropriate linear combinations of other operators g_+', g_-' for which $[\boldsymbol{g}', \boldsymbol{g}'^\dagger]$ is positive definite but not proportional to the identity matrix, one can always define operators g_+, g_- that satisfy $[\boldsymbol{g}, \boldsymbol{g}^\dagger] = \mathbf{1}$.

The four remaining real parameters in g_+ and g_- describe the invariance of the commutation relations (3.2.29) under transformations that take g_+ and g_- into certain independent linear combinations of each other, i.e., under multiplication of the vector $\boldsymbol{g}$ by an arbitrary unitary matrix. Multiplying $\boldsymbol{g}$ by an arbitrary unitary matrix is equivalent to (right-hand) multiplying the operator $\mathbf{U}_g$ of eq. (3.2.25) by a mixing operator $T(q, \chi)$ and two rotation operators $R_\pm(\theta_\pm)$ [eqs. (3.1.44)]. The definition (3.2.27) of $|\boldsymbol{\mu}_g\rangle$ shows that this freedom in the definitions of g_+ and g_- reflects the fact noted in subsection 3.1.4 that a (two-mode) coherent state remains a coherent state when multiplied by rotation and mixing operators [eq. (3.1.49)].

The expressions (3.2.31) for the commutator matrix $[\boldsymbol{g}, \boldsymbol{g}^\dagger]$, together with the matrix properties described above [eqs. (3.2.32) and (3.2.33)], reveal the following simple relations between the noise matrices for a two-mode GPS and the complex matrices P_p, P_x, P_c, and P_s that define operators g_+, g_- unitarily related to a_+, a_- by the same unitary operator:

$$S_x = P_x{}^\dagger P_x\,, \qquad S_p = P_p{}^\dagger P_p\,, \qquad S_{xp} = -\mathrm{Im}(P_x{}^\dagger P_p)\,; \tag{3.2.34a}$$

$$T = -P_c{}^\dagger P_s\,, \tag{3.2.34b}$$

$$Q = \tfrac{1}{2}(P_c{}^\dagger P_c + P_s{}^{\mathrm{T}} P_s{}^*) = P_c{}^\dagger P_c - \tfrac{1}{2}\mathbf{1} = P_s{}^{\mathrm{T}} P_s{}^* + \tfrac{1}{2}\mathbf{1} \tag{3.2.34c}$$

[cf. eqs. (3.2.6), (3.2.9) and (2.2.32)]. These expressions, together with eqs. (3.2.30)–(3.2.33), make more apparent the equalities (3.2.11) satisfied by the noise matrices.

3.2.3. Unitary relation of two-mode GPS to the vacuum state

The form of the unitary operator $\mathbf{U}_g$ in the definition (3.2.27) of the two-mode GPS $|\boldsymbol{\mu}_g\rangle$ is dictated by the transformation (3.2.25) and the forms of g_+ and g_- [eqs. (3.2.19), (3.2.32) and (3.2.33)]. The linearity and absence of any additive constants in the transformation imply that $\mathbf{U}_g = \exp(-iH_g^{(2)}t)$, where $H_g^{(2)}$ is a (Hermitian) linear combination of the ten bilinear products of $a_\pm$ and $a_\pm^\dagger$ ($a_\pm^\dagger a_\pm$, a_+a_-, $a_\pm^2$, $a_+a_-^\dagger$, and their adjoints). That is, the generator $H_g^{(2)}$ of $\mathbf{U}_g$ has the general form $H_0^{(2)} + H_R^{(2)} + H_2^{(2)}$ defined in the Introduction [eqs. (1.1)–(1.4)]. It is shown in subsection 3.3 and appendix A that the operator $\mathbf{U}_g$ can always be written as a product of two single-mode squeeze operators, a two-mode squeeze operator, two rotation operators, and a mixing operator (and an unobservable overall phase factor). The rotation operators can be placed in any position relative to the squeeze and mixing operators, without changing the form of $\mathbf{U}_g$ [eqs. (2.1.25), (3.1.47), (3.1.60)]. When placed to the right of the squeeze operators, they act like the identity operator on the vacuum state and hence are inconsequential. It is not a trivial exercise to commute the mixing operator through the squeeze operators, but it is easy to see that doing so produces a unitary operator whose generator has the form $H_0^{(2)} + H_R^{(2)} + H_2^{(2)}$, multiplied on the right by a mixing operator. This operator, in turn, can be expressed as a product of the three squeeze operators – i.e., an operator like $\mathbf{S}$ of eq. (3.1.71) – multiplied on the right by a rotation and mixing operator. The mixing operator, like the rotation operators, acts like the identity operator on the vacuum state and hence is inconsequential. Hence the unitary operator that relates the most general two-mode GPS to a two-mode coherent state can always be expressed as an operator like $\mathbf{S}$, i.e., a product of the three squeeze operators. The most general two-mode GPS is that defined by eq. (3.2.27) with $\mathbf{U}_g$ equal to $\mathbf{S}$, i.e., it is the state described in eq. (1.18) of the Introduction. A two-mode GPS is thus completely defined by its complex amplitudes μ_+ and μ_- and the values of its six real parameters r, $r_\pm$, φ, and $\varphi_\pm$.

The most general two-mode GPS [eq. (1.18)] is an eigenstate of operators g_+ and g_- defined by the vector relations (3.2.19). The matrices P_c and P_s are given by eqs. (3.1.73b,c):

$$P_c = \begin{pmatrix} \cosh r \cosh r_+ & e^{2i(\varphi-\varphi_-)} \sinh r \sinh r_- \\ e^{-2i(\varphi_+-\varphi)} \sinh r \sinh r_+ & \cosh r \cosh r_- \end{pmatrix},$$
$$P_s = \begin{pmatrix} e^{2i\varphi_+} \cosh r \sinh r_+ & e^{2i\varphi} \sinh r \cosh r \\ e^{2i\varphi} \sinh r \cosh r_+ & e^{2i\varphi_-} \cosh r \sinh r_- \end{pmatrix}. \tag{3.2.35}$$

The matrices P_p and P_x follow from these by the relations (3.2.19b). The complex, symmetric matrices M and Γ are related to the matrices P_c, P_s, P_p, and P_x by eq. (3.2.28). The complex amplitudes μ_+, μ_- and eigenvalues μ_{g+}, μ_{g-} are related to each other through the vector expressions

$$\boldsymbol{\mu} \equiv \langle \boldsymbol{a} \rangle = P_c^\dagger \boldsymbol{\mu}_g - P_s^{\mathrm{T}} \boldsymbol{\mu}_g^*, \qquad \boldsymbol{\mu}_g = P_c \boldsymbol{\mu} + P_s \boldsymbol{\mu}^* \tag{3.2.36}$$

[eqs. (3.2.20) and (3.2.21b)]. The noise matrices S_x, S_p, S_{xp}, T and Q for the two-mode GPS $|\boldsymbol{\mu}_g\rangle$ are obtained by inserting the expressions for P_c, P_s, P_p, and P_x into eqs. (3.2.34). The components of the noise matrices T and Q were given explicitly in the preceding section [eqs. (3.1.85)].

The phase angle δ_x in the coordinate-space wave function $\langle \boldsymbol{x} | \boldsymbol{\mu}_g \rangle$ for the two-mode GPS $|\boldsymbol{\mu}_g\rangle \equiv \mathbf{U}_g |\boldsymbol{\mu}_g\rangle_{\mathrm{coh}}$ can be obtained from eq. (3.2.16). The calculation is described in appendix B. The result is

$$\exp(\tfrac{1}{2} i\delta_x) = (\det P_x^*)^{1/2} / |\det P_x|^{1/2} \tag{3.2.37}$$

[cf. eq. (2.2.37)].

3.2.4. Two-mode momentum-space Gaussian wave function

To conclude this discussion of two-mode Gaussian wave functions, consider briefly the momentum-space wave function for a two-mode Gaussian pure state, $\langle \boldsymbol{p} \mid \boldsymbol{\mu}_g \rangle$ (or $\langle p_+, p_- \mid \boldsymbol{\mu}_g \rangle$), obtained by Fourier transforming $\langle \boldsymbol{x} \mid \boldsymbol{\mu}_g \rangle$ [eq. (3.2.1)]. Here the dimensionless momentum variables p_+ and p_-, components of the column vector $\boldsymbol{p}$, are the eigenvalues of the Hermitian operators $\hat{p}_\pm$. The momentum-space wave function has the following form:

$$\langle \boldsymbol{p} | \boldsymbol{\mu}_g \rangle = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{d}x_+ \, \mathrm{d}x_- \exp(-\mathrm{i}\hat{\boldsymbol{p}}^{\mathrm{T}} \hat{\boldsymbol{x}}) \langle \boldsymbol{x} | \boldsymbol{\mu}_g \rangle$$

$$= \mathcal{N}_g \exp(-\tfrac{1}{2}\mathrm{i}\delta_p) \exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{p}_0{}^{\mathrm{T}}\boldsymbol{x}_0) \exp(-\mathrm{i}\boldsymbol{x}_0{}^{\mathrm{T}}\boldsymbol{p}) \exp(-\tfrac{1}{2}\Delta\boldsymbol{p}^{\mathrm{T}} M^{-1} \Delta\boldsymbol{p}) , \tag{3.2.38a}$$

where the (real) normalization constant $\mathcal{N}_g$ is

$$\mathcal{N}_g = (4\pi^2 \det S_p)^{-1/4} \tag{3.2.38b}$$

[cf. eqs. (3.2.1), (3.2.7), and (2.2.38)]. The phase angle δ_p is related to the coordinate-space phase angle δ_x by

$$\exp(\mathrm{i}\delta_p) = \exp(-\mathrm{i}\delta_x) \frac{\det M}{|\det M|} = -\exp(-\mathrm{i}\delta_x) \frac{\det(S_{xp} + \frac{1}{2}\mathrm{i}\mathbf{1})}{|\det(S_{xp} + \frac{1}{2}\mathrm{i}\mathbf{1})|} . \tag{3.2.39}$$

For the two-mode GPS $|\boldsymbol{\mu}_g\rangle = \mathbf{S}|\boldsymbol{\mu}_g\rangle_{\mathrm{coh}}$ the phase factor $\exp(-\frac{1}{2}\mathrm{i}\delta_p)$ is therefore

$$\exp(-\tfrac{1}{2}\mathrm{i}\delta_p) = (\det P_p{}^*)^{1/2} / |\det P_p|^{1/2} \tag{3.2.40}$$

[cf. eq. (2.2.40)].

The position and momentum probabilities have the usual Gaussian forms:

$$|\langle \boldsymbol{x} | \boldsymbol{\mu}_g \rangle|^2 = (4\pi^2 \det S_x)^{-1/2} \exp(-\tfrac{1}{2}\Delta\boldsymbol{x}^{\mathrm{T}} S_x^{-1} \Delta\boldsymbol{x}) , \tag{3.2.41a}$$

$$|\langle \boldsymbol{p} | \boldsymbol{\mu}_g \rangle|^2 = (4\pi^2 \det S_p)^{-1/2} \exp(-\tfrac{1}{2}\Delta\boldsymbol{p}^{\mathrm{T}} S_p^{-1} \Delta\boldsymbol{p}) \tag{3.2.41b}$$

[cf. eqs. (2.2.41)].

3.3. Four-component vector notation for two-mode GPS

This section describes a four-component vector notation that serves as the basis for an efficient and powerful way of characterizing all two-mode states (pure or mixed) with Gaussian noise statistics. Subsection 3.3.1 defines the fundamental vectors and four-dimensional matrices. Subsection 3.3.2 writes the unitary operators and transformations associated with two-mode GPS in the vector notation. Subsection 3.3.3 discusses the group theoretical significances of the transformation matrices that arise from unitary transformations by the rotation, mixing, and squeeze operators. Subsection 3.3.4 defines four-dimensional second-order noise matrices and discusses some of their important properties. Subsection 3.3.5 uses the vector notation to derive the unitary evolution operator associated with the most general combination of interaction Hamiltonians that can produce a two-mode GPS. This operator is shown to be expressible as the product of two single-mode rotation, displacement, and squeeze operators, a mixing operator, and a two-mode squeeze operator. It is thus proved rigorously that the most general two-mode GPS is unitarily related to a two-mode coherent state by a product of two single-mode squeeze operators and one two-mode squeeze operator.

3.3.1. Fundamental vectors and matrices

The previous discussion has shown that the unitary operators that relate two-mode GPS to the vacuum state and to other two-mode GPS are rotation, mixing, displacement, and squeeze operators. Since these operators induce linear transformations on $a_\pm$ and $a_\pm{}^\dagger$ (or $\hat{x}_\pm$ and $\hat{p}_\pm$), it is useful to define the four-component operator column vectors [33]

$$\mathbf{a} \equiv \begin{pmatrix} a \\ a^* \end{pmatrix}, \qquad \hat{\mathbf{x}} = \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = \mathbf{A}\mathbf{a}, \tag{3.3.1a}$$

$$\mathbf{A} \equiv 2^{-1/2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathrm{i}\mathbf{1} & \mathrm{i}\mathbf{1} \end{pmatrix} = (\mathbf{A}^\dagger)^{-1} \tag{3.3.1b}$$

[cf. eqs. (2.3.1)]. Here and throughout this section the components of four-component vectors are grouped into two two-component vectors, and the components of four-dimensional matrices are grouped into four two-dimensional matrices. The symbol **1** is used to denote both the two- and four-dimensional identity matrices. The expectation values of these operator column vectors are column vectors of complex numbers (for **a**) or real numbers (for $\hat{\mathbf{x}}$):

$$\boldsymbol{\mu} \equiv \langle \mathbf{a} \rangle \equiv \begin{pmatrix} \mu \\ \mu^* \end{pmatrix}, \qquad \boldsymbol{\xi} \equiv \langle \hat{\mathbf{x}} \rangle = \begin{pmatrix} \hat{x}_0 \\ \hat{p}_0 \end{pmatrix} = \mathbf{A}\boldsymbol{\mu}. \tag{3.3.2}$$

The adjoints of the operator column vectors are the row vectors

$$\mathbf{a}^\dagger \equiv (a^\dagger a^{\mathrm{T}}), \qquad \hat{\mathbf{x}}^\dagger = (\hat{x}^{\mathrm{T}} \hat{p}^{\mathrm{T}}) = \hat{\mathbf{x}}^{\mathrm{T}}, \tag{3.3.3}$$

where a superscript "T" means transpose. The transpose of the adjoint of an operator column vector is denoted by a superscript "*":

$$(\mathbf{a}^\dagger)^{\mathrm{T}} \equiv \mathbf{a}^* = \begin{pmatrix} a^* \\ a \end{pmatrix}, \qquad (\hat{\mathbf{x}}^\dagger)^{\mathrm{T}} \equiv \hat{\mathbf{x}}^* = \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = \hat{\mathbf{x}}. \tag{3.3.4}$$

Similar definitions hold for column vectors of complex numbers. Note that the product of a column vector and a row vector, e.g., $\mathbf{a}\mathbf{a}^\dagger$, is a tensor product (i.e., a four-dimensional matrix), whereas the product of a row vector and a column vector, e.g., $\mathbf{a}^\dagger\mathbf{a}$, is a scalar product (i.e., an operator or number).

There are six Hermitian four-dimensional matrices, in addition to the identity matrix, that arise naturally with this vector notation. They are

$$\Sigma_1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \Sigma_2 \equiv \begin{pmatrix} 0 & -\mathrm{i}\mathbf{1} \\ \mathrm{i}\mathbf{1} & 0 \end{pmatrix}, \qquad \Sigma_3 \equiv \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}; \tag{3.3.5a}$$

$$\Gamma_1 \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \Gamma_2 \equiv \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad \Gamma_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}. \tag{3.3.5b}$$

Different, but equivalent, matrices have been used by Milburn [33] to discuss a subset of two-mode GPS. Each of these two sets of matrices satisfies properties analogous to those of the two-dimensional Pauli matrices. For example,

$$\Sigma_i\Sigma_j = \delta_{ij}\mathbf{1} + \mathrm{i}\varepsilon_{ijk}\Sigma_k\,, \qquad \Gamma_i\Gamma_j = \delta_{ij}\mathbf{1} + \mathrm{i}\varepsilon_{ijk}\Gamma_k\,, \qquad i, j, k = 1, 2, 3\,. \tag{3.3.5c}$$

It is useful to define rotated versions of Σ_1, Σ_2 and Γ_1, Γ_2:

$$\Sigma_\varphi \equiv \Sigma_1 \cos 2\varphi - \Sigma_2 \sin 2\varphi = \begin{pmatrix} 0 & \mathrm{e}^{2\mathrm{i}\varphi}\mathbf{1} \\ \mathrm{e}^{-2\mathrm{i}\varphi}\mathbf{1} & 0 \end{pmatrix}, \tag{3.3.6a}$$

$$\Sigma_{\varphi-\pi/4} = \Sigma_1 \sin 2\varphi + \Sigma_2 \cos 2\varphi = \begin{pmatrix} 0 & -\mathrm{i}\mathrm{e}^{2\mathrm{i}\varphi}\mathbf{1} \\ \mathrm{i}\mathrm{e}^{-2\mathrm{i}\varphi}\mathbf{1} & 0 \end{pmatrix}; \tag{3.3.6b}$$

$$\Gamma_\varphi \equiv \Gamma_1 \cos 2\varphi - \Gamma_2 \sin 2\varphi = \begin{pmatrix} \sigma_\varphi & 0 \\ 0 & \sigma_\varphi^* \end{pmatrix}, \tag{3.3.6c}$$

$$\Gamma_{\varphi-\pi/4} \equiv \Gamma_1 \sin 2\varphi + \Gamma_2 \cos 2\varphi = \begin{pmatrix} \sigma_{\varphi-\pi/4} & 0 \\ 0 & \sigma_{\varphi-\pi/4}^* \end{pmatrix} \tag{3.3.6d}$$

[cf. eqs. (2.3.6)]. Note that $[\Sigma_\varphi, \Sigma_{\varphi-\pi/4}] = [\Sigma_1, \Sigma_2] = 2\mathrm{i}\Sigma_3$, and $[\Gamma_\varphi, \Gamma_{\varphi-\pi/4}] = [\Gamma_1, \Gamma_2] = 2\mathrm{i}\Gamma_3$. The following projection matrices are also useful:

$$\tfrac{1}{2}(\mathbf{1} + \Sigma_3\Gamma_3) \equiv P_+\,, \qquad \tfrac{1}{2}(\mathbf{1} - \Sigma_3\Gamma_3) \equiv P_-\,. \tag{3.3.6e}$$

Some of the most useful properties of these matrices follow:

$$[P_\pm, \Sigma_j] = [\Gamma_j, \Sigma_3] = 0\,; \qquad \Sigma_\varphi\Gamma_3\Sigma_\varphi = -\Gamma_3\,; \qquad \Sigma_\varphi\Gamma_\chi\Sigma_\varphi = \Gamma_\chi{}^*\,; \tag{3.3.7a}$$

$$\Sigma_\varphi\Sigma_{\varphi'} = \exp[2\mathrm{i}(\varphi - \varphi')\Sigma_3]\,; \qquad \exp(2\mathrm{i}\varphi'\Sigma_3)\Sigma_\varphi = \Sigma_{\varphi+\varphi'}\,; \tag{3.3.7b}$$

$$\Gamma_\chi\Gamma_{\chi'} = \exp[2\mathrm{i}(\chi - \chi')\Gamma_3]\,; \qquad \exp(2\mathrm{i}\chi'\Gamma_3)\Gamma_\chi = \Gamma_{\chi+\chi'}\,; \tag{3.3.7c}$$

$$\exp(2\mathrm{i}\chi\Gamma_3)\Sigma_\varphi = P_+\Sigma_{\varphi+\chi} + P_-\Sigma_{\varphi-\chi}\,; \tag{3.3.7d}$$

$$P_+\Gamma_\chi\Sigma_\varphi + P_-\Sigma_\varphi\Gamma_\chi = \Sigma_{\varphi+\chi}\Gamma_1\,. \tag{3.3.7e}$$

The commutation relations for $a_\pm$, $a_\pm{}^\dagger$ and $\hat{x}_\pm$, $\hat{p}_\pm$ are contained in the Hermitian commutator matrices

$$[\mathbf{a}, \mathbf{a}^\dagger] \equiv \mathbf{a}\mathbf{a}^\dagger - (\mathbf{a}^*\mathbf{a}^{\mathrm{T}})^{\mathrm{T}} = \Sigma_3\,, \tag{3.3.8a}$$

$$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^{\mathrm{T}}] \equiv \hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}} - (\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}\Sigma_3\mathbf{A}^\dagger = -\Sigma_2 \tag{3.3.8b}$$

[eqs. (3.1.9); cf. eqs. (2.3.7)].

3.3.2. Unitary operators and transformations

The rotation, mixing, displacement, and squeeze operators for two modes are expressed in this vector notation in the following ways:

$$\begin{aligned} &\mathbf{R}(\boldsymbol{\theta}) = \exp[-\mathrm{i}\boldsymbol{a}^\dagger\boldsymbol{\theta}\boldsymbol{a}] = \exp(\mathrm{i}\theta_s)\exp[-\tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger N_\theta\mathbf{a}]\,, \\ &N_\theta \equiv \theta_+P_+ + \theta_-P_- = \theta_s\mathbf{1} + \theta_a\Sigma_3\Gamma_3 = \boldsymbol{\theta}\mathbf{1} \end{aligned} \tag{3.3.9a}$$

[eqs. (3.1.19); cf. eq. (2.3.8a)];

$$D(\boldsymbol{a}, \boldsymbol{\mu}) = \exp[\mathbf{a}^\dagger\Sigma_3\boldsymbol{\mu}] \tag{3.3.9b}$$

[eq. (3.1.29); cf. eq. (2.3.8b)];

$$T(q, \chi) = \exp[-\tfrac{1}{2}\mathrm{i}q\mathbf{a}^\dagger\Gamma_{\chi-\pi/4}\mathbf{a}] \tag{3.3.9c}$$

[eqs. (3.1.36)];

$$\begin{aligned} &S_{1+}(r_+, \varphi_+)S_{1-}(r_-, \varphi_-) = \exp[-\tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger N_1\mathbf{a}]\,, \\ &N_1 \equiv r_+P_+\Sigma_{\varphi_+-\pi/4} + r_-P_-\Sigma_{\varphi_--\pi/4} \end{aligned} \tag{3.3.9d}$$

[eq. (3.1.50); cf. eq. (2.3.8c)];

$$\begin{aligned} &S(r, \varphi) = \exp[-\tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger N_2\mathbf{a}]\,, \\ &N_2 \equiv r\Sigma_{\varphi-\pi/4}\Gamma_1 = \begin{pmatrix} 0 & -\mathrm{i}r\,\mathrm{e}^{2\mathrm{i}\varphi}\sigma_1 \\ \mathrm{i}r\,\mathrm{e}^{-2\mathrm{i}\varphi}\sigma_1 & 0 \end{pmatrix} \end{aligned} \tag{3.3.9e}$$

[eqs. (3.1.53)].

A unitary transformation by the two-mode displacement operator on the components of $\mathbf{a}$ or $\hat{\mathbf{x}}$ results in the addition of a constant column vector:

$$D(\boldsymbol{a}, \boldsymbol{\mu})\mathbf{a}D^\dagger(\boldsymbol{a}, \boldsymbol{\mu}) = \mathbf{a} - \boldsymbol{\mu}\,, \qquad D(\boldsymbol{a}, \boldsymbol{\mu})\hat{\mathbf{x}}D^\dagger(\boldsymbol{a}, \boldsymbol{\mu}) = \hat{\mathbf{x}} - \boldsymbol{\xi} \tag{3.3.10}$$

[eq. (3.1.31); cf. eq. (3.3.9)]. Unitary transformations by rotation, mixing, and squeeze operators result in matrix transformations of $\mathbf{a}$ and $\hat{\mathbf{x}}$. One way to obtain these transformation matrices is to note the following general relation, which follows from the commutator matrix $[\mathbf{a}, \mathbf{a}^\dagger] = \Sigma_3$: Let K be an arbitrary four-dimensional, symmetric matrix,

$$K \equiv \begin{pmatrix} K_a & K_b \\ K_c & K_d \end{pmatrix}, \tag{3.3.11a}$$

where K_a, K_b, K_c, and K_d are arbitrary two-dimensional matrices. Then

$$[\mathbf{a}^\dagger K \mathbf{a}, \mathbf{a}] = K_0 \mathbf{a}\,,$$
$$K_0 \equiv -\Sigma_3(K + \Sigma_1 K^{\mathrm{T}} \Sigma_1) = \begin{pmatrix} -(K_a + K_d^{\mathrm{T}}) & -(K_b + K_b^{\mathrm{T}}) \\ (K_c + K_c^{\mathrm{T}}) & (K_a^{\mathrm{T}} + K_d) \end{pmatrix}. \tag{3.3.11b}$$

This implies that

$$\exp(\mathbf{a}^\dagger K \mathbf{a})\mathbf{a} \exp(-\mathbf{a}^\dagger K \mathbf{a}) = \mathrm{e}^{K_0}\mathbf{a}\,. \tag{3.3.11c}$$

Note that if K_a, K_b, K_c, and K_d are all symmetric,

$$K_0 = -(K_a + K_d)\Sigma_3 + [K, \Sigma_3] \tag{3.3.11d}$$

[cf. eqs. (2.3.10)]. For the matrices K in eqs. (3.3.9), K_b and K_c are symmetric and $K_d = K_a^{\mathrm{T}}$; hence

$$K_0 = -2\Sigma_3 K\,. \tag{3.3.11e}$$

The matrix transformations induced on the column vectors $\mathbf{a}$ and $\hat{\mathbf{x}}$ by the rotation operator $\mathbf{R}(\boldsymbol{\theta})$ are therefore

$$\mathbf{R}(\boldsymbol{\theta})\mathbf{a}\mathbf{R}^\dagger(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{a}(\boldsymbol{\theta}) \\ \boldsymbol{a}^*(\boldsymbol{\theta}) \end{pmatrix} = \exp(\mathrm{i}N_\theta \Sigma_3)\mathbf{a} \equiv \mathbf{a}(\boldsymbol{\theta})\,. \tag{3.3.12a}$$

$$\mathbf{R}(\boldsymbol{\theta})\hat{\mathbf{x}}\mathbf{R}^\dagger(\boldsymbol{\theta}) = \begin{pmatrix} \hat{\boldsymbol{x}}(\boldsymbol{\theta}) \\ \hat{\boldsymbol{p}}(\boldsymbol{\theta}) \end{pmatrix} = \mathbf{A}\exp(\mathrm{i}N_\theta \Sigma_3)\mathbf{A}^\dagger \hat{\mathbf{x}} = \exp(-\mathrm{i}N_\theta \Sigma_2)\hat{\mathbf{x}} \equiv \hat{\mathbf{x}}(\boldsymbol{\theta})\,; \tag{3.3.12b}$$

$$\exp(\mathrm{i}N_\theta \Sigma_3) = P_+ \exp(\mathrm{i}\theta_+ \Sigma_3) + P_- \exp(\mathrm{i}\theta_- \Sigma_3) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{\theta}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}} \end{pmatrix}, \tag{3.3.12c}$$

$$\exp(-\mathrm{i}N_\theta \Sigma_2) = P_+(\cos\theta_+ \mathbf{1} - \mathrm{i}\sin\theta_+ \Sigma_2) + P_-(\cos\theta_- \mathbf{1} - \mathrm{i}\sin\theta_- \Sigma_2)$$
$$= \begin{pmatrix} C & -S \\ S & C \end{pmatrix}, \qquad C \equiv \begin{pmatrix} \cos\theta_+ & 0 \\ 0 & \cos\theta_- \end{pmatrix}, \qquad S \equiv \begin{pmatrix} \sin\theta_+ & 0 \\ 0 & \sin\theta_- \end{pmatrix} \tag{3.3.12d}$$

[eqs. (3.1.24); cf. eqs. (2.3.11)].

The matrix transformation induced on the column vector $\mathbf{a}$ by the mixing operator $T(q, \chi)$ is

$$T(q, \chi)\mathbf{a}T^\dagger(q, \chi) = \exp(\mathrm{i}q\Sigma_3 \Gamma_{\chi - \pi/4})\mathbf{a} \tag{3.3.13}$$

[eqs. (3.1.38)]. A product of two single-mode squeeze operators induces the matrix transformation

$$S_{1+}(r_+, \varphi_+)S_{1-}(r_-, \varphi_-)\mathbf{a}S_{1-}^{\dagger}(r_-, \varphi_-)S_{1+}^{\dagger}(r_+, \varphi_+) = P_1\mathbf{a}\,,$$
$$P_1 \equiv \exp(\mathrm{i}\Sigma_3 N_1) = P_+ \exp(r_+ \Sigma_{\varphi_-}) + P_- \exp(r_- \Sigma_{\varphi_-}) = \begin{pmatrix} P_{1\mathrm{c}} & P_{1\mathrm{s}} \\ P_{1\mathrm{s}}^* & P_{1\mathrm{c}} \end{pmatrix} = P_1^\dagger \tag{3.3.14}$$

[eqs. (3.1.52); cf. eqs. (2.3.12)]. Finally, the matrix transformation induced on **a** by a two-mode squeeze operator is

$$S(r,\varphi)\mathbf{a}S^\dagger(r,\varphi)=P_2\mathbf{a}\,,\qquad P_2\equiv\exp(\mathrm{i}\Sigma_3N_2)=\exp(r\Sigma_\varphi\Gamma_1)=\begin{pmatrix}\cosh r\,\mathbf{1} & \mathrm{e}^{2\mathrm{i}\varphi}\sinh r\,\sigma_1\\ \mathrm{e}^{-2\mathrm{i}\varphi}\sinh r\,\sigma_1 & \cosh r\,\mathbf{1}\end{pmatrix} \tag{3.3.15}$$

[eq. (3.1.55b)]. The simple form of the transformation matrix P_2 associated with the two-mode squeeze operator $S(r,\varphi)$ or, equivalently, the simple form of the matrix N_2 that defines $S(r,\varphi)$ [eq. (3.3.9e)], makes it possible to describe all the properties of two-mode squeezed states with a two-component vector notation. That two-component vector notation is one which naturally groups a_+ with $a_-^\dagger$ and a_- with $a_+^\dagger$ [see the discussion in subsection 3.1.5c, eqs. (3.1.68)–(3.1.70), and ref. [20]]. The transformation matrix that results from unitarily transforming **a** with the product $S_{1+}S_{1-}S\equiv\mathbf{S}$ of three squeeze operators is equal to the product of the transformation matrices (3.3.14) and (3.3.15), and is denoted by the symbol **P**:

$$\mathbf{S}\mathbf{a}\mathbf{S}^\dagger=\mathbf{P}\mathbf{a}\,,\qquad \mathbf{P}\equiv P_2P_1=\begin{pmatrix}P_\mathrm{c} & P_\mathrm{s}\\ P_\mathrm{s}^* & P_\mathrm{c}^*\end{pmatrix} \tag{3.3.16}$$

[eqs. (3.1.73)]. Transformation matrices for the column vector $\hat{\mathbf{x}}$ are unitarily related to those for **a** by the unitary matrix **A** [eq. (3.3.1b)].

3.3.3. Group theoretical properties of transformation matrices

The transformation matrices (3.3.12)–(3.3.16) arise naturally, without specific reference to the rotation, mixing, or squeeze operators, from the requirement that a unitary transformation on a_+ and a_- (or $\hat{x}_+$, $\hat{x}_-$, $\hat{p}_+$, and $\hat{p}_-$) preserve the commutators (3.3.8). Consider, for example, the real, four-dimensional matrices $\bar{M}$ that describe transformations induced on the components of the real column vector $\hat{\mathbf{x}}$ by a unitary operator U: $\bar{M}\hat{\mathbf{x}}\equiv U\hat{\mathbf{x}}U^\dagger$. The unitarity of U implies that the matrices $\bar{M}$ preserve the antisymmetric commutator matrix $[\hat{\mathbf{x}},\hat{\mathbf{x}}^\mathrm{T}]=-\Sigma_2$ [eq. (3.3.8b)]:

$$\bar{M}\Sigma_2\bar{M}^\mathrm{T}=\Sigma_2=\bar{M}^\mathrm{T}\Sigma_2\bar{M} \tag{3.3.17a}$$

[cf. eq. (2.3.18a)]. The real matrices $\bar{M}$ that satisfy this condition have unity determinant. They comprise the ten-parameter symplectic group Sp(4, R) [41]. Milburn [33] has used the properties of Sp(4, R) to pick out special two-mode GPS that comprise a subset of the set of two-mode minimum-uncertainty states defined in this paper (see discussion in subsection 3.1.7). Transformations induced on the components of the column vector $\mathbf{a}=\mathbf{A}^\dagger\hat{\mathbf{x}}$ by the same unitary operator U are described by complex, four-dimensional matrices M, $M\mathbf{a}\equiv U\mathbf{a}U^\dagger$. The matrices M are unitarily related to the real matrices $\bar{M}$ through the matrix **A** [eq. (3.3.1b)]:

$$M=\mathbf{A}^\dagger\bar{M}\mathbf{A}\,. \tag{3.3.17b}$$

The matrices M have unity determinant and preserve the Hermitian commutator matrix $[\mathbf{a},\mathbf{a}^\dagger]=\Sigma_3$ [eq. (3.3.8a)]:

$$M\Sigma_3M^\dagger=\Sigma_3=M^\dagger\Sigma_3M\,. \tag{3.3.17c}$$

They comprise a ten-parameter subgroup of the 15-parameter, noncompact Lie group SU(2, 2) [41], isomorphic to Sp(4, R).

The linearity of these matrix transformations implies that the unitary operator U is an exponential of the ten bilinear combinations of the annihilation and creation operators for the two modes. The ten real parameters that characterize the transformation matrices M and $\bar{M}$ are thus related to the ten parameters of the unitary operator, i.e., to the coefficients of the ten bilinear combinations. The most general such unitary operator can be expressed as a product of a rotation and mixing operator, two single-mode squeeze operators, and one two-mode squeeze operator,

$$U = \mathbf{S}T(q, \chi)\mathbf{R}(\boldsymbol{\theta}) \tag{3.3.17d}$$

(proof in appendix A). Hence, from the preceding discussion of the rotation, mixing, and squeeze operators, the transformation matrices M have the general form

$$M = \exp(\mathrm{i}N_\theta\Sigma_3)\exp(\mathrm{i}q\Sigma_3\Gamma_{\chi-\pi/4})\mathbf{P}\,, \tag{3.3.17e}$$

where $r_\pm$, r, $\varphi_\pm$, φ, q, χ, and $\theta_\pm$ are real, continuous parameters [eqs. (3.3.12)–(3.3.16)].

It is instructive to obtain the general form (3.3.17e) for the matrices M in another way. Begin by noting that any four-dimensional matrix M that describes a linear transformation on the components of the column vector $\mathbf{a}$ must satisfy

$$M^* = \Sigma_1 M \Sigma_1\,, \tag{3.3.18a}$$

since $\mathbf{a} = \Sigma_1\mathbf{a}^*$. This implies that the matrix M has the general form

$$M = \begin{pmatrix} M_a & M_b \\ M_b{}^* & M_a{}^* \end{pmatrix}, \tag{3.3.18b}$$

where M_a and M_b are two-dimensional complex matrices. It also implies the following equality:

$$M\Sigma_3 M^\dagger \Sigma_3 = M\Sigma_2 M^{\mathrm{T}}\Sigma_2 \tag{3.3.18c}$$

[cf. eqs. (2.3.19)]. Because the matrix M describes a unitary transformation on the components of $\mathbf{a}$, it preserves both the Hermitian commutator matrix $[\mathbf{a}, \mathbf{a}^\dagger] = \Sigma_3$ [eq. (3.3.17c)] and the antisymmetric commutator matrix $[\mathbf{a}, \mathbf{a}^{\mathrm{T}}] = \mathrm{i}\Sigma_2$; i.e., both expressions in eq. (3.3.18c) are equal to the identity matrix. Unitarity thus determines six of the sixteen real parameters associated with the matrix M of eq. (3.3.18b), by imposing the following equivalent sets of conditions on the two-dimensional matrices M_a and M_b:

$$M_a M_a{}^\dagger - M_b M_b{}^\dagger = \mathbf{1}\,, \qquad M_a M_b{}^{\mathrm{T}} - (M_a M_b{}^{\mathrm{T}})^{\mathrm{T}} = 0 \tag{3.3.18d}$$

[imposed by $M\Sigma_3 M^\dagger\Sigma_3 = M\Sigma_2 M^{\mathrm{T}}\Sigma_2 = \mathbf{1}$], or

$$M_a{}^\dagger M_a - M_b{}^{\mathrm{T}} M_b{}^* = \mathbf{1}\,, \qquad M_a{}^\dagger M_b - (M_a{}^\dagger M_b)^{\mathrm{T}} = 0 \tag{3.3.18e}$$

[imposed by $M^{\dagger}\Sigma_3 M\Sigma_3 = M^{\mathrm{T}}\Sigma_2 M\Sigma_2 = \mathbf{1}$]. Note that these conditions also ensure that $\det M = 1$. The matrices M that describe unitary transformations on the components of the column vector $\mathbf{a}$ therefore have the general form (3.3.18b), subject to the conditions (3.3.18d) or (3.3.18e) – i.e., they have the general form (3.3.17e).

3.3.4. Second-order noise matrices

The four-component vector notation is very useful for calculating second-order noise moments of $a_{\pm}$, $a_{\pm}{}^{\dagger}$, $\hat{x}_{\pm}$, and $\hat{p}_{\pm}$ – i.e., the noise matrices Q, T, S_x, S_p, and S_{xp}. The four-dimensional matrix that contains all second-order noise moments of $a_{\pm}$ and $a_{\pm}{}^{\dagger}$ is the Hermitian matrix

$$\mathcal{Q} \equiv \langle \Delta\mathbf{a}\,\Delta\mathbf{a}^{\dagger}\rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}(\langle \Delta\mathbf{a}\,\Delta\mathbf{a}^{\dagger}\rangle + \langle \Delta\mathbf{a}^{*}\,\Delta\mathbf{a}^{\mathrm{T}}\rangle^{\mathrm{T}}) = \begin{pmatrix} Q & T \\ T^{*} & Q^{*} \end{pmatrix} = \mathcal{Q}^{\dagger} \tag{3.3.19a}$$

[cf. eq. (2.3.20)]. The four-dimensional matrix that contains all second-order noise moments of $\hat{x}_{\pm}$ and $\hat{p}_{\pm}$ is the real, symmetric covariance matrix

$$\begin{aligned}\mathcal{S} &\equiv \langle \Delta\hat{\mathbf{x}}\,\Delta\hat{\mathbf{x}}^{\mathrm{T}}\rangle_{\mathrm{sym}} \equiv \tfrac{1}{2}(\langle \Delta\hat{\mathbf{x}}\,\Delta\hat{\mathbf{x}}^{\mathrm{T}}\rangle + \langle \Delta\hat{\mathbf{x}}\,\Delta\hat{\mathbf{x}}^{\mathrm{T}}\rangle^{\mathrm{T}}) \\ &= \begin{pmatrix} S_x & S_{xp} \\ S_{xp}{}^{\mathrm{T}} & S_p \end{pmatrix} = A\mathcal{Q}A^{\dagger} = \mathcal{S}^{*} = \mathcal{S}^{\mathrm{T}}\end{aligned} \tag{3.3.19b}$$

[cf. eq. (2.3.21)]. The relations (3.2.11) imply that for two-mode GPS these matrices satisfy

$$\mathcal{Q}\Sigma_3\mathcal{Q}\Sigma_3 = \tfrac{1}{4}\mathbf{1}\,, \tag{3.3.20a}$$

$$\mathcal{S}\Sigma_2\mathcal{S}\Sigma_2 = \tfrac{1}{4}\mathbf{1} \tag{3.3.20b}$$

[cf. eqs. (2.3.22)]. Hence their determinants are both equal to $\frac{1}{4}$. For a two-mode coherent state, both are proportional to the identity matrix:

$$\mathcal{Q}_{\mathrm{coh}} = \mathcal{S}_{\mathrm{coh}} = \tfrac{1}{2}\mathbf{1} \tag{3.3.21}$$

[eqs. (3.1.91)].

The noise matrix $\mathcal{Q}$ for a two-mode state $|\Psi\rangle$ is related to that of the rotated state $\mathbf{R}(\boldsymbol{\theta})|\Psi\rangle$ in the following way:

$$\begin{aligned}\langle \mathbf{R}^{\dagger}(\boldsymbol{\theta})(\Delta\mathbf{a}\,\Delta\mathbf{a}^{\dagger})_{\mathrm{sym}}\mathbf{R}(\boldsymbol{\theta})\rangle &= \langle \Delta\mathbf{a}(-\boldsymbol{\theta})\,\Delta\mathbf{a}^{\dagger}(-\boldsymbol{\theta})\rangle_{\mathrm{sym}} \\ &= \exp(-\mathrm{i}N_{\theta}\Sigma_3)\mathcal{Q}\exp(\mathrm{i}N_{\theta}\Sigma_3) \equiv \mathcal{Q}(-\boldsymbol{\theta})\end{aligned} \tag{3.3.22}$$

[eqs. (3.1.28)]. It is related to that of the transformed state $T(q,\chi)|\Psi\rangle$ by

$$\langle T^{\dagger}(q,\chi)(\Delta\mathbf{a}\,\Delta\mathbf{a}^{\dagger})_{\mathrm{sym}}T(q,\chi)\rangle = \exp(-\mathrm{i}q\Sigma_3\Gamma_{\chi-\pi/4})\mathcal{Q}\exp(\mathrm{i}q\Sigma_3\Gamma_{\chi-\pi/4})\,. \tag{3.3.23}$$

The noise matrix $\mathcal{Q}$ for a two-mode state $|\Psi\rangle$ is related to that of the transformed state $\mathbf{S}|\Psi\rangle$ by

$$\langle \mathbf{S}^{\dagger}(\Delta\mathbf{a}\,\Delta\mathbf{a}^{\dagger})_{\mathrm{sym}}\mathbf{S}\rangle = \mathbf{P}^{-1}\mathcal{Q}(\mathbf{P}^{-1})^{\dagger}\,. \tag{3.3.24}$$

This immediately tells one, for example, that the noise matrix $\varnothing$ for the most general two-mode GPS $|\boldsymbol{\mu}_g\rangle \equiv \mathbf{S}|\boldsymbol{\mu}_g\rangle_{\text{coh}}$ [eq. (3.2.27)] is

$$\varnothing = \tfrac{1}{2}(\mathbf{P}^{\dagger}\mathbf{P})^{-1} = \tfrac{1}{2}\Sigma_3\mathbf{P}^{\dagger}\mathbf{P}\Sigma_3 \tag{3.3.25}$$

[eq. (3.3.21); cf. eqs. (3.1.85), (3.2.34b,c)].

3.3.5. Evolution operator for general two-mode GPS

Finally, the four-component vector notation is helpful for seeing how the unitary operator whose generator is an arbitrary combination of the Hermitian forms $H_{\mathrm{R}}^{(2)}$, $H_1^{(2)}$, and $H_2^{(2)}$ factors into a product of three squeeze operators, a mixing operator, a (two-mode) rotation operator, and a (two-mode) displacement operator (and an overall phase factor). By giving these generators arbitrary time dependences, one can calculate the evolution operator associated with the most general combination of Hamiltonians that can produce two-mode GPS. This result is given here, with supporting details presented in appendix A.

The photon number-conserving Hamiltonians associated with two-mode GPS are expressed in vector notation by

$$H_{\mathrm{R}+}^{(1)}(t) + H_{\mathrm{R}-}^{(1)}(t) = -\omega_{\mathrm{s}} + \tfrac{1}{2}\mathbf{a}^{\dagger}N_{\omega}\mathbf{a}\,, \qquad N_{\omega} \equiv \omega_+ P_+ + \omega_- P_- = \omega_{\mathrm{s}}\mathbf{1} + \omega_{\mathrm{a}}\Sigma_3\Gamma_3\,,$$

$$\omega_{\pm} \equiv \omega_{\mathrm{s}} \pm \omega_{\mathrm{a}}\,, \qquad \omega_{\substack{\mathrm{s}\\ \mathrm{a}}} \equiv \tfrac{1}{2}(\omega \pm \omega_-)\,; \tag{3.3.26a}$$

$$H_{\mathrm{R}+-}(t) = \tfrac{1}{2}\rho\,\mathbf{a}^{\dagger}\Gamma_{\chi_\rho - \pi/4}\mathbf{a}\,, \tag{3.3.26b}$$

where $\omega_{\pm}$ (or ω_{s}, ω_{c}), ρ, and χ_ρ are real-valued functions of time t [eqs. (1.5), (2.6), (3.3.9a,c)]. The linear Hamiltonian associated with two-mode GPS has the form

$$H_1^{(2)}(t) = \mathrm{i}\mathbf{a}^{\dagger}\Sigma_3\boldsymbol{\lambda}\,, \qquad \boldsymbol{\lambda} \equiv \begin{pmatrix}\boldsymbol{\lambda}\\ \boldsymbol{\lambda}^*\end{pmatrix}, \qquad \boldsymbol{\lambda} \equiv \begin{pmatrix}\lambda_+\\ \lambda_-\end{pmatrix}, \tag{3.3.26c}$$

where λ_+ and λ_- are complex-valued functions of time t [eqs. (1.4a), (3.3.9b)]. The quadratic, non-photon-number-conserving Hamiltonians associated with two-mode GPS are

$$H_{2+}^{(1)}(t) + H_{2-}^{(1)}(t) = \tfrac{1}{2}\mathbf{a}^{\dagger}(\kappa_+ P_+ \Sigma_{\varphi_{\kappa_+} - \pi/4} + \kappa_- P_- \Sigma_{\varphi_{\kappa_-} - \pi/4})\mathbf{a}\,, \tag{3.3.26d}$$

where $\kappa_{\pm}$ and $\varphi_{\kappa_\pm}$ are real-valued functions of t [eqs. (1.8), (3.3.9d)], and

$$H_{2+-}(t) = \tfrac{1}{2}\kappa\,\mathbf{a}^{\dagger}\Sigma_{\varphi_\kappa - \pi/4}\Gamma_1\mathbf{a}\,, \tag{3.3.26e}$$

where κ and φ are real-valued functions of time t [eqs. (1.7), (3.3.9e)].

The evolution operator $U(t)$ is the solution to the equation

$$\mathrm{i}\partial_t U(t) = H_g^{(2)}(t)U(t)\,, \qquad U(0) = 1\,,$$

$$H_g^{(2)} \equiv [H_{\mathrm{R}+}^{(1)} + H_{\mathrm{R}-}^{(1)} + H_{\mathrm{R}+-} + H_1^{(2)} + H_{2+}^{(1)} + H_{2-}^{(1)} + H_{2+-}]\,. \tag{3.3.27}$$

It can be written as a product (in any order) of the three squeeze operators, a mixing, rotation, and a (two-mode) displacement operator, and an overall phase factor. For illustration, consider the following two forms for $U(t)$:

$$U(t) = e^{i\delta} D(\mathbf{a}, \boldsymbol{\mu}) S_{1+}(r_+, \varphi_+) S_{1-}(r_-, \varphi_-) S(r, \varphi)\, T(q, \chi) \mathbf{R}(\boldsymbol{\theta}) \tag{3.3.28a}$$

$$= e^{i\delta} S_{1+}(r_+, \varphi_+) S_{1-}(r_-, \varphi_-) S(r, \varphi) T(q, \chi) \mathbf{R}(\boldsymbol{\theta}) D(\mathbf{a}, \boldsymbol{\mu}_g) \tag{3.3.28b}$$

[cf. eq. (3.2.27)]. Here δ, $r_\pm$, r, $\varphi_\pm$, φ, q, χ, and $\theta_\pm$ are real-valued functions of time, and $\mu_\pm$ and $\mu_{g\pm}$ are complex-valued functions of time. For notational convenience, throughout the remainder of this section the product of the three squeeze operators is denoted by $\mathbf{S}$ [eq. (3.1.71)], and the parameters of the other unitary operators are omitted, i.e., $T \equiv T(q, \chi)$ and $\mathbf{R} \equiv \mathbf{R}(\boldsymbol{\theta})$. The state $U(t)|0\rangle$ is an eigenstate of operators $g_\pm = U(t) a_\pm U^\dagger(t)$ (with eigenvalues $\mu_{g\pm}$), whose relations to $a_\pm$ are described by the vector relation

$$\mathbf{g} \equiv \begin{pmatrix} g \\ g^* \end{pmatrix} = \mathbf{S} T \mathbf{R} \mathbf{a} \mathbf{R}^\dagger T^\dagger \mathbf{S}^\dagger = \exp(iN_\theta \Sigma_3) \exp(iq\Sigma_3 \Gamma_{\chi - \pi/4}) \mathbf{P} \mathbf{a} \tag{3.3.29a}$$

[eqs. (3.3.12)–(3.3.16)]. The complex eigenvalues $\mu_{g\pm}$ are therefore related to the complex amplitudes $\mu_\pm \equiv \langle a_\pm \rangle$ by

$$\boldsymbol{\mu}_g \equiv \begin{pmatrix} \mu_g \\ \mu_g^* \end{pmatrix} = \exp(iN_\theta \Sigma_3) \exp(iq\Sigma_3 \Gamma_{\chi - \pi/4}) \mathbf{P} \boldsymbol{\mu}\,. \tag{3.3.29b}$$

It is shown in appendix A that the relations of the functions r, $r_\pm$, φ, $\varphi_\pm$, $\theta_\pm$ (or θ_s, θ_a), $\mu_{g\pm}$ (or $\mu_\pm$), and δ to the Hamiltonian functions κ, $\kappa_\pm$, φ_κ, $\varphi_{\kappa\pm}$, $\omega_\pm$ (or ω_s, ω_a), and $\lambda_\pm$ take the form of matrix, vector, and scalar equalities. The vector and scalar equalities are

$$\dot{\boldsymbol{\mu}}_g = \exp(iN_\theta \Sigma_3) \exp(iq\Sigma_3 \Gamma_{\chi - \pi/4} \mathbf{P} \boldsymbol{\lambda} \equiv \boldsymbol{\lambda}_g\,, \tag{3.3.30a}$$

$$\dot{\delta} + \dot{\theta}_s - \omega_s = -\tfrac{1}{2} i \boldsymbol{\mu}_g^\dagger \Sigma_3 \dot{\boldsymbol{\mu}}_g = \mathrm{Im}(\mu_+^* \lambda_+ + \mu_-^* \lambda_-) = \mathrm{Im}(\boldsymbol{\mu}_g^\dagger \boldsymbol{\lambda}_g) = \mathrm{Im}(\boldsymbol{\mu}^\dagger \boldsymbol{\lambda}) \tag{3.3.30b}$$

[eq. (3.3.17c); cf. eqs. (2.3.32b,c)]. (Dots denote derivatives with respect to time.) The matrix equality is given in its full generality in eqs. (A.15)–(A.17)]. The initial conditions, dictated by $U(0) = 1$, are

$$\delta(0) = r(0) = r_\pm(0) = q(0) = \theta_\pm(0) = \mu_{g\pm}(0) = \mu_\pm(0) = 0\,. \tag{3.3.31}$$

For illustration, consider the case where (i)

$$\varphi_{\kappa\pm} = \varphi_\kappa \pm \chi_\rho\,, \tag{3.3.32a}$$

and (ii) the time dependences of these parameters are

$$\varphi_{\kappa\pm} \equiv \varphi_{\kappa\pm 0} - \int_0^t \omega_\pm \, \mathrm{d}t \,, \tag{3.3.32b}$$

where $\varphi_{\kappa\pm 0}$ are constants. The matrix equality then implies that

$$\varphi_\pm = \varphi_{\kappa\pm} \,, \qquad \varphi = \varphi_\kappa \,, \qquad \chi = \chi_\rho \,, \qquad \theta_\pm = \int_0^t \omega_\pm \, \mathrm{d}t \,; \tag{3.3.33a}$$

$$\begin{gathered} \rho = \dot{q} \cosh 2r \cosh 2r_\mathrm{a} - \dot{r} \sinh 2r_\mathrm{a} \,, \\ \kappa = \dot{r} \cosh 2r_\mathrm{a} - \dot{q} \cosh 2r \sinh 2r_\mathrm{a} \,, \qquad \kappa_\pm = \dot{r}_\pm \pm \dot{q} \sinh 2r \,, \end{gathered} \tag{3.3.33b}$$

where $r_\pm \equiv r_\mathrm{s} \pm r_\mathrm{a}$. If the mixing interaction is absent ($\rho = 0$), eqs. (3.3.33b) reduce to the following coupled equations for r, $r_\pm$ (or r_s, r_a), and q:

$$r \operatorname{sech} 2r_\mathrm{a} = \kappa \,, \qquad \dot{r}_\pm \pm \kappa \sinh 2r_\mathrm{a} \tanh 2r = \kappa_\pm \,, \qquad \dot{q} \cosh 2r / \sinh 2r_\mathrm{a} = \kappa \,. \tag{3.3.34a}$$

If $\kappa_+ = \kappa_- \equiv \kappa'$, these coupled equations have the simple solutions

$$r_+ = r_- = \int_0^t \kappa' \, \mathrm{d}t \,, \qquad r = \int_0^t \kappa \, \mathrm{d}t \,, \qquad q = 0 \,. \tag{3.3.34b}$$

The phase angle δ and complex amplitudes $\mu_\pm$ (or eigenvalues $\mu_{g\pm}$) are obtained by using the solutions for r, $r_\pm$, q, φ, $\varphi_\pm$, and χ (from the matrix equality) to solve the vector and scalar equalities (3.3.30a,b).

4. Concluding remarks

Nature abounds with phenomena whose classical or quantum mechanical description involves fields or states that are Gaussian in nature. The reasoning, formalism, and results presented here form a basis for a simple but complete description of all such states and fields. This paper has focused on pure states, defined as specific unitary operators acting on the vacuum state. Its goal has been to provide a description of two-mode Gaussian pure states that is as complete and useful as the existing description of single-mode Gaussian pure states. The motivation has come primarily from the desire for a realistic description of "two-photon" devices, such as parametric amplifiers, which are capable of producing states with exceptional low-noise properties, compared with the coherent states produced by conventional one-photon devices such as lasers. Two-photon devices operate by producing pairs of correlated photons. In the special case of a degenerate device, these photons occupy a single mode; in general, however the photons occupy different modes, and they may differ from each other in frequency. The (pure) states produced by ideal degenerate and nondegenerate two-photon devices are single-mode and two-mode squeezed states, respectively. The properties and importance of two-mode squeezed states were discussed only briefly here, since they are the subject of detailed discussions elsewhere [18–20]. Here, their origin and properties were compared with those of other two-mode Gaussian pure states, e.g., products of two single-mode squeezed states, produced by two degenerate two-photon devices.

Although this paper has focused on pure states, the results and formalism described in it serve as a basis for a general description of all states that are mixtures of Gaussian pure states (Gaussian mixed states). The analog for mixed states of a wave function is a quasi-probability distribution [52] (QPD), defined as a $2N$-dimensional Fourier transform of a characteristic function [32]. The characteristic function of an N-mode state described by a density operator ρ is defined as the trace of the product of ρ and an (appropriately ordered) N-mode displacement operator. Thus, an N-mode Gaussian mixed state, a mixture of N single-mode Gaussian pure states, is a mixed state whose QPD (or characteristic function) is Gaussian (i.e., an exponential of complex-valued linear and quadratic forms in "position" and "momentum" variables). Characteristic functions and QPDs have been used successfully in quantum optics, for example, to provide realistic descriptions (including losses and other nonideal effects) of devices that produce single-mode GPS and mixtures thereof – e.g., "one-photon" devices such as the laser [10, 11, 29] and degenerate two-photon devices [53, 54]. More recently, the author has defined special "two-mode" characteristic functions and QPDs (based on the electric-field quadrature-phase amplitudes) that are specifically suited to describing realistic nondegenerate two-photon devices [18, 55]. More general two-mode characteristic functions and QPDs, based on the formalism described in this paper, are the keys to a realistic description of all devices that produce (mixtures of) states whose noise statistics are Gaussian in the sense defined here.

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Appendix A: Evolution operators for GPS

In this appendix certain properties of the unitary operators associated with single-mode and two-mode GPS are derived that are useful for calculating the general evolution operators described in subsections 2.3 and 3.3 [eqs. (2.3.28)–(2.3.36), eqs. (3.3.26)–(3.3.34)].

A1. Single-mode GPS

One way to find the single-mode evolution operator $U(t)$ defined by eqs. (2.3.28)–(2.3.30) is to take the derivative with respect to time of either of the factored expressions (2.3.30) for $U(t)$, and match terms with the Hamiltonian. This is the approach described here. An alternative approach, which permits calculation of everything except the phase factor $e^{i\delta}$ in the expressions (2.3.30) for $U(t)$, is to solve in the Heisenberg picture the matrix equation $\boldsymbol{a}_s(t) = M(t)\boldsymbol{a}_s(0) + \boldsymbol{\mu}_s(t)$, and identify the unitary operator U_g that generates M; then $\boldsymbol{a}_s(t) = D^\dagger(a,\mu)U_g^\dagger \boldsymbol{a}_s(0) D(a,\mu)U_g$ [29]. The evolution operator $U(t)$ is equal, up to an overall phase factor, to the product of U_g and the displacement operator $D(a,\mu)$.

The first (and hardest) task involved with computing the time derivative of the expressions (2.3.30) for $U(t)$ is to compute the (first-order) derivatives of each of the unitary operators $S_1(r,\varphi)$, $R(\theta)$, and $D(a,\mu)$; the second task is to commute these operators through each other. The time derivative of the second expression for $U(t)$ in eq. (2.3.30) is

$$\dot{U}U^{-1} = \mathrm{i}\dot{\delta} + \dot{S}_1 S_1^\dagger + S_1(\dot{R}R^\dagger)S_1^\dagger + S_1 R(\dot{D}_g D_g^\dagger)R^\dagger S_1^\dagger \,, \tag{A.1}$$

where $D_g \equiv D(a,\mu_g)$, and a superposed dot denotes a single derivative with respect to time. These time derivatives can be found using the general rule

$$[\partial_t \, \mathrm{e}^{f(t)}]\, \mathrm{e}^{-f(t)} = \sum_{n=0}^{\infty} \frac{\{f^n \dot{f}\}}{(n+1)!} \equiv \dot{f} + \frac{1}{2!}[f,\dot{f}] + \frac{1}{3!}[f,[f,\dot{f}]] + \cdots \tag{A.2}$$

(derived in subsection A3 of this appendix). The time derivative of $S_1(r,\varphi)$ can be calculated with the help of the following facts:

$$\dot{\sigma}_\varphi = -2\dot{\varphi}\sigma_{\varphi-\pi/4}\,, \quad \dot{\sigma}_{\varphi-\pi/4} = 2\dot{\varphi}\sigma_\varphi\,; \tag{A.3a}$$

$$[\boldsymbol{a}_s^\dagger \sigma_{\varphi-\pi/4}\boldsymbol{a}_s, \boldsymbol{a}_s^\dagger \sigma_\varphi \boldsymbol{a}_s] = 4\mathrm{i}\boldsymbol{a}_s^\dagger \boldsymbol{a}_s\,, \tag{A.3b}$$

$$[\boldsymbol{a}_s^\dagger \sigma_{\varphi-\pi/4}\boldsymbol{a}_s, \boldsymbol{a}_s^\dagger \boldsymbol{a}_s] = 4\mathrm{i}\boldsymbol{a}_s^\dagger \sigma_\varphi \boldsymbol{a}_s\,. \tag{A.3c}$$

The result is

$$\dot{S}_1 S_1^\dagger = -\tfrac{1}{2}\mathrm{i}\boldsymbol{a}_s^\dagger[-\dot{\varphi}\mathbf{1} + \dot{\varphi}C_{2r,\varphi} + \dot{r}\sigma_{\varphi-\pi/4}]\boldsymbol{a}_s\,. \tag{A.4}$$

The time derivatives of $R(\theta)$ and $D(a,\mu_g)$ are

$$\dot{R}R^\dagger = -\tfrac{1}{2}\mathrm{i}\dot{\theta}(\boldsymbol{a}_s^\dagger \boldsymbol{a}_s - 1)\,, \tag{A.5}$$

$$\dot{D}_g D_g^\dagger = (\boldsymbol{a}_s^\dagger - \tfrac{1}{2}\boldsymbol{\mu}_{gs}^\dagger)\sigma_3 \dot{\boldsymbol{\mu}}_{gs}\,. \tag{A.6}$$

Note that $\dot{\mu}_g \equiv \partial_t(\mu_g) \neq (\dot{\mu})_g$.

Commuting the operators through each other to find the last two terms in eq. (A.1) is accomplished using the transformations in eqs. (2.3.11) and (2.3.12). Equating $\mathrm{i}\dot{U}U^\dagger$ to the sum of the Hamiltonians on the right-hand side of eq. (2.3.29) then results in the relations (2.3.32), which define the functions r, φ, θ, μ_g (or μ), and δ uniquely in terms of the Hamiltonian functions κ, φ_κ, Ω, and λ.

A2. Two-mode GPS

As in the single-mode case, the first task is to calculate the (first-order) time derivatives of the various unitary operators, and the second is to commute the operators through each other. The time derivative of the expression (3.3.28b) for $U(t)$ is

$$\dot{U}U^\dagger = \mathrm{i}\dot{\delta} + \dot{\mathbf{S}}\mathbf{S}^\dagger + \mathbf{S}(\dot{T}T^\dagger)\mathbf{S}^\dagger + \mathbf{S}T(\dot{\mathbf{R}}\mathbf{R}^\dagger)T^\dagger\mathbf{S}^\dagger + \mathbf{S}T\mathbf{R}(\dot{D}_g D_g{}^\dagger)\mathbf{R}^\dagger T^\dagger \mathbf{S}^\dagger\,, \tag{A.7}$$

where $D_g \equiv D(\boldsymbol{a}, \boldsymbol{\mu}_g)$, $\mathbf{S} \equiv S_{1+}S_{1-}S$, and a superposed dot denotes a single derivative with respect to time. The time derivatives of these operators are found using the formula (A.2). For the rotation and displacement operators the calculation is straightforward, giving

$$\dot{\mathbf{R}}\mathbf{R}^\dagger = \mathrm{i}\dot{\theta}_\mathrm{s} - \tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger N_\theta \mathbf{a}, \qquad N_{\dot\theta} \equiv \dot{\theta}_+ P_+ + \dot{\theta}_- P_- = \dot{\theta}_\mathrm{s}\mathbf{1} + \dot{\theta}_\mathrm{a}\Sigma_3\Gamma_3\,; \tag{A.8}$$

$$\dot{D}_g D_g{}^\dagger = (\mathbf{a}^\dagger - \tfrac{1}{2}\boldsymbol{\mu}_g^\dagger)\Sigma_3\dot{\boldsymbol{\mu}}_g\,. \tag{A.9}$$

For the remaining operators it is useful to have the analogs of eqs. (A.3). For the mixing operator these are

$$\dot{\Gamma}_\chi = -2\dot{\chi}\Gamma_{\chi-\pi/4}\,, \qquad \dot{\Gamma}_{\chi-\pi/4} = 2\dot{\chi}\Gamma_\chi\,; \tag{A.10a}$$

$$[\mathbf{a}^\dagger\Gamma_{\chi-\pi/4}\mathbf{a}, \mathbf{a}^\dagger\Gamma_\chi\mathbf{a}] = -4\mathrm{i}\,\mathbf{a}^\dagger\Sigma_3\Gamma_3\mathbf{a}\,, \tag{A.10b}$$

$$[\mathbf{a}^\dagger\Gamma_{\chi-\pi/4}\mathbf{a}, \mathbf{a}^\dagger\Sigma_3\Gamma_3\mathbf{a}] = 4\mathrm{i}\,\mathbf{a}^\dagger\Gamma_\chi\mathbf{a}\,. \tag{A.10c}$$

These lead easily to the result

$$\dot{T}T^\dagger = -\tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger M_T\mathbf{a}\,, \qquad M_T \equiv -\dot{\chi}\Sigma_3\Gamma_3 + \dot{\chi}\Sigma_3\Gamma_3\exp(2\mathrm{i}q\Sigma_3\Gamma_{\chi-\pi/4}) + \dot{q}\Gamma_{\chi-\pi/4}\,. \tag{A.11}$$

For the three squeeze operators the relevant facts are

$$\dot{\Sigma}_\varphi = -2\dot{\varphi}\Sigma_{\varphi-\pi/4}\,, \qquad \dot{\Sigma}_{\varphi-\pi/4} = 2\dot{\varphi}\Sigma_\varphi\,; \tag{A.12a}$$

$$[\mathbf{a}^\dagger\Sigma_{\varphi-\pi/4}\mathbf{a}, \mathbf{a}^\dagger\Sigma_\varphi\mathbf{a}] = 4\mathrm{i}\mathbf{a}^\dagger\mathbf{a}\,, \tag{A.12b}$$

$$[\mathbf{a}^\dagger\Sigma_{\varphi-\pi/4}\mathbf{a}, \mathbf{a}^\dagger\mathbf{a}] = 4\mathrm{i}\,\mathbf{a}^\dagger\Sigma_\varphi\mathbf{a}\,. \tag{A.12c}$$

These lead to the results

$$(\partial_t S_{1+}S_{1-})(S_{1+}S_{1-})^\dagger = -\tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger M_1\mathbf{a}\,,$$

$$M_1 \equiv -\dot{\varphi}_+P_+ - \dot{\varphi}_-P_- + \dot{\varphi}_+P_+\exp(2r_+\Sigma_{\varphi+}) + \dot{\varphi}_-P_-\exp(2r_-\Sigma_{\varphi-}) + \dot{r}_+P_+\Sigma_{\varphi+-\pi/4} + \dot{r}_-P_-\Sigma_{\varphi--\pi/4}\,; \tag{A.13}$$

$$\dot{S}S^\dagger = -\tfrac{1}{2}\mathrm{i}\mathbf{a}^\dagger M_2\mathbf{a}\,, \qquad M_2 \equiv -\dot{\varphi}\mathbf{1} + \dot{\varphi}\exp(2r\Sigma_\varphi\Gamma_1) + \dot{r}\Sigma_{\varphi-\pi/4}\Gamma_1\,. \tag{A.14}$$

Commuting the operators through each other to find the last three terms in eq. (A.7) can be accomplished with the help of the transformations described in eqs. (3.3.12)–(3.3.16). Equating $\mathrm{i}\dot{U}U^\dagger$ to the sum of the Hamiltonians on the right-hand side of eq. (3.3.27) then results in matrix, vector, and scalar equalities, which define the functions r, $r_\pm$, φ, $\varphi_\pm$, q, χ, $\theta_\pm$, $\mu_{g\pm}$ (or $\mu_\pm$), and δ uniquely in terms of the Hamiltonian functions κ, $\kappa_\pm$, φ_κ, $\varphi_{\kappa\pm}$, ρ, χ_ρ, $\Omega_\pm$, and $\lambda_\pm$. The vector and scalar equalities were given in eqs. (3.3.30). The matrix equality is

$$\begin{aligned} &M_1 + P_1 M_2 P_1 + \mathbf{P}^\dagger M_T \mathbf{P} + \mathbf{P}^\dagger \exp(-\mathrm{i}q\Sigma_3\Gamma_{\chi-\pi/4}) N_{\dot\theta} \exp(\mathrm{i}q\Sigma_3\Gamma_{\chi-\pi/4})\mathbf{P} \\ &\quad = N_\omega + \rho\Gamma_{\chi_\rho-\pi/4} + \kappa_+ P_+ \Sigma_{\varphi_{\kappa_+}-\pi/4} + \kappa_- P_- \Sigma_{\varphi_{\kappa_-}-\pi/4} + \kappa\Sigma_{\varphi_\kappa-\pi/4}\Gamma_1 \end{aligned} \tag{A.15}$$

[eqs. (3.3.12)–(3.3.16) and (A.8)–(A.14)]. The matrix transformations required in order to put the left-hand side of eq. (A.15) into a form that is easily compared with the right-hand side (the Hamiltonian) can be accomplished fairly easily by making use of the properties of the matrices Σ_i and Γ_i noted in eqs. (3.3.5)–(3.3.7). The terms that comprise the left-hand side of eq. (A.15) are listed below, with the four-dimensional matrix that multiplies it listed at the left of each term. The following shorthand notations are used:

$$\varphi_\pm - \varphi \mp \chi \equiv \beta_\pm\,, \qquad \varphi + \theta_s \equiv \gamma_s\,, \qquad \chi + \theta_a \equiv \gamma_a\,. \tag{A.16}$$

The terms are as follows:

$$\begin{aligned} P_+:\quad & -\dot\varphi_+ + \cosh 2r_+[\dot\beta_+ + \dot\gamma_s\cosh 2r + \dot\gamma_a\cos 2q] \\ & +\sinh 2r\sinh 2r_+[\dot\gamma_a\sin 2q\cos 2\beta_+ - \dot q\sin 2\beta_+]\,, \end{aligned} \tag{A.17a}$$

$$P_-:\quad -\dot\varphi_- + \cosh 2r_-[\dot\beta_- + \dot\gamma_s\cosh 2r - \dot\gamma_a\cos 2q] + \sinh 2r\sinh 2r_-[\dot\gamma_a\sin 2q\cos 2\beta_- + \dot q\sin 2\beta_-]\,, \tag{A.17b}$$

$$\begin{aligned} &\Gamma_{\chi-\pi/4}:\quad \dot q\cosh 2r\cosh r_+\cosh r_-\,, \\ &\Gamma_{\beta_+-\beta_-+\chi-\pi/4}:\quad \dot q\cosh 2r\sinh r_+\sinh r_-\,, \\ &\Gamma_{\beta_++\chi-\pi/4}:\quad -\dot r\sinh r_+\cosh r_-\,, \\ &\Gamma_{\chi-\beta_--\pi/4}:\quad \dot r\sinh r_-\cosh r_+\,; \end{aligned} \tag{A.17c}$$

$$\begin{aligned} &P_+\Sigma_{\varphi_+-\pi/4}:\quad \dot r_+\,, \\ &P_+\Sigma_{\varphi_++\beta_+-\pi/4}:\quad -\dot q\sinh 2r\sinh^2 r_+\,, \\ &P_+\Sigma_{\varphi_+-\beta_+-\pi/4}:\quad \dot q\sinh 2r\cosh^2 r_+\,; \end{aligned} \tag{A.17d}$$

$$\begin{aligned} &P_-\Sigma_{\varphi_--\pi/4}:\quad \dot r_-\,, \\ &P_-\Sigma_{\varphi_-+\beta_--\pi/4}:\quad \dot q\sinh 2r\sinh^2 r_-\,, \\ &P_-\Sigma_{\varphi_--\beta_--\pi/4}:\quad -\dot q\sinh 2r\cosh^2 r_-\,, \end{aligned} \tag{A.17e}$$

$$\Sigma_{\varphi-\pi/4}\Gamma_1:\quad \dot{r}\cosh r_+\cosh r_- ,$$

$$\Sigma_{\varphi+\beta_++\beta_--\pi/4}\Gamma_1:\quad -\dot{r}\sinh r_+\sinh r_- ,$$

$$\Sigma_{\varphi+\beta_+-\pi/4}:\quad -\dot{q}\cosh 2r\sinh r_+\cosh r_- ,$$

$$\Sigma_{\varphi+\beta_--\pi/4}:\quad \dot{q}\cosh 2r\sinh r_-\cosh r_+ ; \tag{A.17f}$$

$$\Sigma_{\varphi}\Gamma_1:\quad \dot{\gamma}_s\sinh 2r\cosh r_+\cosh r_- ,$$

$$\Sigma_{\varphi+\beta_++\beta_-}\Gamma_1:\quad \dot{\gamma}_s\sinh 2r\sinh r_+\sinh r_- ,$$

$$\Sigma_{\varphi+\beta_+}\Gamma_1:\quad \dot{\gamma}_a\sin 2q\cosh 2r\sinh r_+\cosh r_- ,$$

$$\Sigma_{\varphi+\beta_-}\Gamma_1:\quad \dot{\gamma}_a\sin 2q\cosh 2r\sinh r_-\cosh r_+ ; \tag{A.17g}$$

$$\Gamma_{\chi}:\quad \dot{\gamma}_a\sin 2q\cosh 2r\cosh r_+\cosh r_- ,$$

$$\Gamma_{\chi+\beta_+-\beta_-}:\quad \dot{\gamma}_a\sin 2q\cosh 2r\sinh r_+\sinh r_- ,$$

$$\Gamma_{\chi+\beta_+}:\quad \dot{\gamma}_s\sinh 2r\sinh r_+\cosh r_- ,$$

$$\Gamma_{\chi-\beta_-}:\quad \dot{\gamma}_s\sinh 2r\sinh r_-\cosh r_+ ; \tag{A.17h}$$

$$P_+\Sigma_{\varphi_+}:\quad \sinh 2r_+[\dot{\beta}_+ + \dot{\gamma}_s\cosh 2r + \dot{\gamma}_a\cos 2q] ,$$

$$P_+\Sigma_{\varphi_++\beta_+}:\quad \dot{\gamma}_a\sin 2q\sinh 2r\sinh^2 r_+ ,$$

$$P_+\Sigma_{\varphi_+-\beta_+}:\quad \dot{\gamma}_a\sin 2q\sinh 2r\cosh^2 r_+ ; \tag{A.17i}$$

$$P_-\Sigma_{\varphi_-}:\quad \sinh 2r_-[\dot{\beta}_- + \dot{\gamma}_s\cosh 2r - \dot{\gamma}_a\cos 2q] ,$$

$$P_-\Sigma_{\varphi_-+\beta_-}:\quad \dot{\gamma}_a\sin 2q\sinh 2r\sinh^2 r_- ,$$

$$P_-\Sigma_{\varphi_--\beta_-}:\quad \dot{\gamma}_a\sin 2q\sinh 2r\cosh^2 r_- . \tag{A.17j}$$

The obvious simplifying case is that considered in subsection 3.3 [eqs. (3.3.32)], for which

$$\beta_+ = \beta_- = \dot{\gamma}_s = \dot{\gamma}_a = 0 . \tag{A.18}$$

A3. Derivation of eq. (A.2)

The formula (A.2) for $\partial_t\, e^{f(t)}$ can be found as follows: First, use the standard definition for derivatives, and keep only terms of lowest order in Δt:

$$\partial_t\, e^{f(t)} = \lim_{\Delta t\to 0}\left(\frac{e^{f(t+\Delta t)} - e^{f(t)}}{\Delta t}\right) = \lim_{\Delta t\to 0}\left(\frac{e^{f(t)+\Delta t\dot{f}(t)} - e^{f(t)}}{\Delta t}\right). \tag{A.19}$$

Next, use the definition of e^x to write

$$\exp(f+\Delta t\dot{f}) = \lim_{n\to\infty}\left[e^f + \sum_{j=0}^{n-1}\left(1+\frac{f}{n}\right)^j\left(\frac{\Delta t\dot{f}}{n}\right)\left(1+\frac{f}{n}\right)^{n-j-1} + O(\Delta t)^2\right]. \tag{A.20}$$

As $n\to\infty$, let $1/n\to dx$, where the variable x $(=j/n)$ runs from 0 to 1. Then

$$e^{f+\Delta t\dot{f}} - e^f = \Delta t\int_0^1 dx\, e^{xf}\dot{f}\, e^{-xf}\, e^f + O(\Delta t)^2. \tag{A.21}$$

The relation (A.2) is then proved by noting that

$$e^{xf}\dot{f}\, e^{-xf} = \sum_0^\infty \frac{x^n}{n!}\{f^n\dot{f}\} \equiv \dot{f} + x[f,\dot{f}] + \frac{x^2}{2!}[f,[f,\dot{f}]] + \cdots. \tag{A.22}$$

Appendix B: Phase factors for wave functions

The phase factors $\exp(\frac{1}{2}i\delta_x)$ and $\exp(\frac{1}{2}i\delta_p)$ in the single-mode and two-mode coordinate- and momentum-space wave functions can be found using eqs. (2.2.16) and (3.2.16), respectively. Two methods are described here for single-mode GPS (i.e., single-mode squeezed states): one uses a factored form for the single-mode squeeze operator [eq. (2.3.15)]; the other uses a differential equation. For the most general two-mode GPS the calculation is more challenging. Factored forms for the product of the three squeeze operators are not as convenient for this calculation as their single-mode counterparts, because they involve more than one operator that does not act like the identity operator on the vacuum state; furthermore, their derivation is nontrivial. The differential equation approach, while possibly more promising, still involves a painful process of commuting operators through each other. Although no rigorous derivation is given here, the phase factor for a general two-mode GPS can be guessed with reasonable certainty. Of course, for two-mode squeezed states the calculation is no more difficult than for a single-mode squeezed state, since the two-mode squeeze operator factors just as easily. Similarly, for states that are a product of two single-mode squeezed states the phase factor is just the product of the single-mode phase factors.

The coordinate-space phase factor $\exp(\frac{1}{2}i\delta_x)$ for a single-mode squeezed state is found, from eq. (2.2.16), by calculating

$$\langle x=0|S_1(r,\varphi)|0\rangle\,. \tag{B.1}$$

The factored form (2.3.15) for $S_1(r,\varphi)$ implies that

$$S_1(r,\varphi)|0\rangle = (\cosh r)^{-1/2}\sum_{n=0}^\infty \frac{(-\frac{1}{2}e^{2i\varphi}\tanh r)^n}{n!}\sqrt{(2n)!}|2n\rangle\,. \tag{B.2}$$

The wave function for a number state $|n\rangle \equiv [(n)!]^{-1/2}(a^\dagger)^n|0\rangle$ is [17]

$$\langle x \mid n\rangle = (2^n n!)^{-1/4} H_n(x)\langle x \mid 0\rangle\,, \tag{B.3a}$$

where

$$\langle x|0\rangle = \pi^{-1/4}\exp(-\tfrac{1}{2}x^2)\,, \tag{B.3b}$$

and $H_n(x)$ is a Hermite polynomial with the property that

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!!\,. \tag{B.3c}$$

These relations then imply that

$$\begin{aligned}\langle x=0|S_1(r,\varphi)|0\rangle &= \pi^{-1/4}(\cosh r)^{-1/2}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!}(\tfrac{1}{2}\mathrm{e}^{2\mathrm{i}\varphi}\tanh r)^n \\ &= \pi^{-1/4}(\cosh r - \mathrm{e}^{2\mathrm{i}\varphi}\sinh r)^{-1/2}\,.\end{aligned} \tag{B.4}$$

Hence the phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ in the coordinate-space wave function of a single-mode squeezed state $|\mu_\alpha\rangle_{(r,\varphi)}$ is

$$\exp(\tfrac{1}{2}\mathrm{i}\delta_x) = \frac{(\cosh r - \mathrm{e}^{-2\mathrm{i}\varphi}\sinh r)^{1/2}}{|\cosh r - \mathrm{e}^{-2\mathrm{i}\varphi}\sinh r|^{1/2}} = \frac{(\rho_x{}^*)^{1/2}}{|\rho_x|^{1/2}}\,, \tag{B.5}$$

as given in eq. (2.2.37).

The phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ for a single-mode squeezed state can also be calculated using a differential equation. One considers the wave function $\langle x=0|S_1(r,\varphi)|0\rangle \equiv f_0(r)$ as a function of the squeeze factor r. Differentiating it, and replacing the operators a and $a^\dagger$ with the appropriate linear combinations of $\hat{x}$ and $\hat{p}$, leads to a simple first-order differential equation in r, whose solution is

$$\begin{aligned}f_0(r) &\equiv N_g\exp(\tfrac{1}{2}\mathrm{i}\delta_x) \equiv \pi^{-1/4}|\cosh r - \mathrm{e}^{2\mathrm{i}\varphi}\sinh r|^{-1/2}\exp(\tfrac{1}{2}\mathrm{i}\delta_x) \\ &= \pi^{-1/4}(\cosh r - \mathrm{e}^{2\mathrm{i}\varphi}\sinh r)^{-1/2}\end{aligned} \tag{B.6}$$

[eqs. (2.1.35a), ((2.2.7)].

The phase factors in the wave functions of two-mode squeezed states can be calculated in these same ways; the first method is somewhat simpler, since the required factored form for the two-mode squeeze operator is already known from the single-mode result (see discussion in subsection 3.1.5c). The calculation parallels that given in eqs. (B.2)–(B.5) above. The result is the following expression for the phase factor $\exp(\frac{1}{2}\mathrm{i}\delta_x)$ in the wave function of a two-mode squeezed state $|\boldsymbol{\mu}_\alpha\rangle_{(r,\varphi)}$:

$$\exp(\tfrac{1}{2}\mathrm{i}\delta_x) = (\cosh^2 r - \mathrm{e}^{-4\mathrm{i}\varphi}\sinh^2 r)^{1/2}/|\cosh^2 r - \mathrm{e}^{-4\mathrm{i}\varphi}\sinh^2 r|^{1/2}\,. \tag{B.7}$$

The phase factor in the wave function of a product of two single-mode squeezed states is just the product of the phase factors for two single-mode squeeze states [eq. (B.5)]. These results suggest strongly that the phase factor for the coordinate-space wave function for the general two-mode GPS $|\boldsymbol{\mu}_g\rangle \equiv S_{1+}S_{1-}S|\boldsymbol{\mu}\rangle_{\rm coh}$ is given by the following expression:

$$\exp(\tfrac{1}{2}\mathrm{i}\delta_x) = (\det P_x{}^*)^{1/2}/|\det P_x|^{1/2} \tag{B.8}$$

[eq. (3.2.37)].

Appendix C: Simultaneous eigenstates of complex operators

The complex (non-Hermitian) operators of which Gaussian states are eigenstates are linear combinations of creation and annihilation operators. The discussions in subsections 2.2 and 3.2 showed that a single operator g of this type has a complete (or overcomplete) set of normalizable eigenstates if and only if the commutator $[g, g^\dagger]$ is a positive real number, since this is equivalent to the condition that the wave function be normalizable (recall that $[g, g^\dagger] = 0$ corresponds to a delta-function wave function).

Two-mode Gaussian pure states are eigenstates of two linearly independent complex operators g_+ and g_-, each of which is a linear combination of the creation and annihilation operators for the two modes. The discussion in subsection 3.2 showed that two such operators have a common, complete (or overcomplete) set of normalizable eigenstates if and only if (i) the commutator $[g_+, g_-] = 0$, and (ii) the Hermitian commutator matrix Y_g is positive definite, where

$$Y_g \equiv [\boldsymbol{g}, \boldsymbol{g}^\dagger] = \begin{pmatrix} [g_+, g_+{}^\dagger] & [g_+, g_-{}^\dagger] \\ [g_-, g_+{}^\dagger] & [g_-, g_-{}^\dagger] \end{pmatrix} = Y_g^\dagger , \qquad \boldsymbol{g} \equiv \begin{pmatrix} g_+ \\ g_- \end{pmatrix} . \tag{C.1}$$

These conditions are equivalent to the requirement that the two-mode Gaussian wave function be normalizable. That g_+ and g_- must commute with each other in order for them to have a complete set of simultaneous eigenstates is obvious. The further requirement that Y_g be positive definite – but not necessarily diagonal – is not so obvious. One might have expected, incorrectly, that the two operators must commute completely, i.e., that the commutator $[g_+, g_-{}^\dagger]$ must also vanish. Following is a simple argument that shows why Y_g must be positive definite, but not necessarily diagonal.

Let g_+ and g_- be two complex operators (with c-number commutators) that commute with each other completely,

$$[g_+, g_-] = [g, g_-{}^\dagger] = 0 . \tag{C.2}$$

Suppose also that each has a complete (or overcomplete) set of normalizable eigenstates, i.e., $[g_\pm, g_\pm{}^\dagger] > 0$. Clearly there exist normalizable states that are eigenstates of both g_+ and g_-, and hence also of all linear combinations of g_+ and g_-. Consider two such (independent) linear combinations, g_+' and g_-', defined by

$$\boldsymbol{g}' \equiv \begin{pmatrix} g_+' \\ g_-' \end{pmatrix} = K\boldsymbol{g} , \tag{C.3}$$

where K is any two-dimensional nonsingular matrix ($\det K \neq 0$). The operators g_+' and g_-' will certainly commute with each other, $[g_+', g_-'] = 0$, but the commutator $[g_+', g_-'^\dagger]$ will not, in general, be zero. The commutator $[g_+', g_-'^\dagger]$ will vanish if, for example, the operators g_+' and g_-' are obtained by unitarily transforming g_+ and g_- by the same unitary operator U, $g_\pm' \equiv U g_\pm U^\dagger$, since in that case all commutators are preserved. If, however, the operators g_+' and g_-' are obtained by unitarily transforming g_+ and g_- with different unitary operators,

$$g_\pm' \equiv U_\pm g_\pm U_\pm^{\dagger}, \tag{C.4}$$

then only the commutators $[g_\pm, g_\pm^\dagger]$ must be preserved, and the commutator $[g_+', g_-'^\dagger]$ need not vanish. The Hermitian commutator matrix $Y_{g'}$ for the operators g_+' and g_-' is related to that of g_+ and g_- by

$$Y_{g'} \equiv [g', g'^\dagger] = K Y_g K^\dagger . \tag{C.5}$$

In general, all elements of $Y_{g'}$ can differ from those of Y_g. The property of the commutator matrix that is preserved in the transformation (C.5) is positive definiteness. A Hermitian matrix is positive definite if and only if its eigenvalues are positive; i.e., if $\boldsymbol{\mu}$ is a vector of complex numbers with components μ_i, $i = 1, 2, \dots, N$ and M is an N-dimensional matrix with components M_{ij}, then M is positive definite if and only if

$$\boldsymbol{\mu}^\dagger M \boldsymbol{\mu} = M_{ij}\mu_i{}^* \mu_j > 0 \tag{C.6}$$

for all vectors $\boldsymbol{\mu}$. This shows clearly that $Y_{g'}$ is positive definite if and only if Y_g is positive definite, since

$$\boldsymbol{\mu}^\dagger Y_g \boldsymbol{\mu} = (K\boldsymbol{\mu})^\dagger Y_{g'} (K\boldsymbol{\mu}) \geq 0 . \tag{C.7}$$

Appendix D: Squeezing in general two-mode GPS

This appendix presents some supporting mathematical details for the discussion of squeezing in subsection 1.8 of the Introduction.

Consider the electric-field operator for a collection of modes with frequencies between $\Omega - \varepsilon$ and $\Omega + \varepsilon$, all travelling in the x-direction with identical polarizations. It has the form

$$E(x, t) = f \int_{\Omega-\varepsilon}^{\Omega+\varepsilon} \mathrm{d}\omega\, \omega^{1/2} [a(\omega)\,\mathrm{e}^{-\mathrm{i}\omega u} + a^+(\omega)\,\mathrm{e}^{\mathrm{i}\omega u}]$$

$$= f \int_0^{\varepsilon} \mathrm{d}\varepsilon\, \{\mathrm{e}^{-\mathrm{i}\varepsilon u}[(\Omega+\varepsilon)^{1/2} a_+(\varepsilon)\,\mathrm{e}^{-\mathrm{i}\Omega u} + (\Omega-\varepsilon)^{1/2} a_-{}^\dagger(\varepsilon)\,\mathrm{e}^{\mathrm{i}\Omega u}] + \text{h.c.}\} \tag{D.1a}$$

$$\equiv E_1(x, t) \cos \Omega u + E_2(x, t) \sin \Omega u , \tag{D.1b}$$

where $u \equiv t - x$, $a_\pm(\varepsilon) \equiv a(\Omega \pm \varepsilon)$ are annihilation operators for the modes, f is a spatially dependent proportionality factor, and "h.c." means "Hermitian conjugate". The quadrature phases E_1 and E_2 describe modulation of the carrier waves $\cos \Omega u$ and $\sin \Omega u$. Their Fourier components at frequencies ε are proportional to the quadrature-phase amplitudes defined in eqs. (1.21):

$$E_j(x, t) = f(2\Omega)^{1/2} \int_0^{\varepsilon} \mathrm{d}\varepsilon\, [\alpha_j(\varepsilon)\, \mathrm{e}^{-\mathrm{i}\varepsilon u} + \alpha_j^{\dagger}(\varepsilon)\, \mathrm{e}^{\mathrm{i}\varepsilon u}] \tag{D.2a}$$

$$= 2f\Omega^{1/2} \int_0^{\varepsilon} \mathrm{d}\varepsilon\, [\alpha_{j1}(\varepsilon) \cos \varepsilon u + \alpha_{j2}(\varepsilon) \sin \varepsilon u], \qquad j = 1, 2, \tag{D.2b}$$

where $\alpha_{j1}(\varepsilon)$ and $\alpha_{j2}(\varepsilon)$ are ($2^{1/2}$ times) the real and imaginary parts of α_j, respectively:

$$\alpha_j(\varepsilon) \equiv 2^{-1/2}[\alpha_{j1}(\varepsilon) + \mathrm{i}\, \alpha_{j2}(\varepsilon)]\,. \tag{D.3}$$

The variance of $E_j(x, t)$ is

$$\langle(\Delta E_j)^2\rangle \propto \int_0^{\varepsilon} \int_0^{\varepsilon'} \mathrm{d}\varepsilon\, \mathrm{d}\varepsilon'\, [\langle \Delta\alpha_j(\varepsilon)\, \Delta\alpha_j(\varepsilon')\rangle \exp[-\mathrm{i}(\varepsilon + \varepsilon')u] + \text{h.c.}$$
$$+ 2\langle \Delta\alpha_j(\varepsilon)\, \Delta\alpha_j^{\dagger}(\varepsilon')\rangle_{\text{sym}} \exp[-\mathrm{i}(\varepsilon - \varepsilon')u]]\,. \tag{D.4}$$

Restrict attention now to (i) Gaussian fields, for which the moments of E and E_j obey a Gaussian distribution and hence are characterized by second-order moments, and (ii) fields for which the modulations at different frequencies ε are independent of each other, so that each of the noise moments in the above expression vanishes when $\varepsilon \neq \varepsilon'$. Such fields can be viewed quantum mechanically as a collection of independent two-mode GPS, all of which have the form of eq. (1.18). One need therefore consider only one of these two-mode states, i.e., a single modulation frequency ε. The filtered output of a heterodyne detector (the frequency ε component of E_j) represents just such a state. When this output is mixed with a wave $\cos \varepsilon u$, the resulting dc signal is proportional to $\langle \alpha_{j1}(\varepsilon)\rangle$, [eqs. (D.2) and (D.3)]. The noise in this signal is proportional to the variance of α_{j1}. For the general two-mode GPS $|\boldsymbol{\mu}_g\rangle$ [eq. (1.18)], the mean-square uncertainties in α_1 and α_2 and the second-order noise moments $\langle(\Delta\alpha_1)^2\rangle$ and $\langle(\Delta\alpha_2)^2\rangle$ are

$$\langle|\Delta\alpha_{\substack{1\\2}}|^2\rangle = \tfrac{1}{2}\{\cosh 2r(1 + \lambda_+^2 \sinh^2 r_+ + \lambda_-^2 \sinh^2 r_-)$$
$$\mp \lambda_+\lambda_- \sinh 2r[(\cosh r_+ \cosh r_- \cos 2\varphi + \sinh r_+ \sinh r_- \cos 2(\varphi - \varphi_+ - \varphi_-)]\}\,, \tag{D.5a}$$

$$\langle(\Delta\alpha_{\substack{1\\2}})^2\rangle = \tfrac{1}{2}[\lambda_+\lambda_- \sinh 2r(\exp[2\mathrm{i}(\varphi - \varphi_-)] \cosh r_+ \sinh r_- + \exp[-2\mathrm{i}(\varphi - \varphi_+)] \cosh r_- \sinh r_+)$$
$$\mp \tfrac{1}{2} \cosh 2r(\lambda_+^2 \exp(2\mathrm{i}\varphi_+) \sinh 2r_+ + \lambda_-^2 \exp(-2\mathrm{i}\varphi_-) \sinh 2r_-)]\,, \tag{D.5b}$$

$$\lambda_{\pm} \equiv (1 \pm \varepsilon/\Omega)^{1/2} \tag{D.5c}$$

[eq. (3.1.85)]. The variances of α_{j1} and α_{j2} are related to these by

$$\langle(\Delta\alpha_{j\,\frac{1}{2}})^2\rangle = \langle|\Delta\alpha_j|^2\rangle \pm \mathrm{Re}(\langle(\Delta\alpha_j)^2\rangle)\,. \tag{D.6}$$

It is clear from eq. (D.5a) that the mean-square uncertainty in α_j, which constitutes the time-independent contribution to the variance of E_j, can be smaller than the coherent-state value of $\frac{1}{2}$ only if $r \neq 0$, as stated in the Introduction. For a two-mode squeezed state ($r_+ = r_- = 0$), the noise moments $\langle(\Delta\alpha_j)^2\rangle$ vanish (TSQP noise), which implies that the variances of α_{j1} and α_{j2} are equal; the modulation signal at the output of the heterodyne detector has random-phase noise. Consider now states that do not exhibit TSQP noise, i.e., states for which the modulation signal has phase-sensitive noise. In particular, let us look for states in which the variance of α_{11}, say, can be smaller than it is in either a coherent state or a two-mode squeezed state. Clearly such a state must have $\varphi_+ = \varphi_- = \varphi = 0$; i.e., it is a two-mode MUS, as defined in subsection 3.1.7. For two-mode MUS, the variances of α_{j1} and α_{j2} are

$$\langle(\Delta\alpha_{\,\substack{11\\21}})^2\rangle = \tfrac{1}{2}[\tfrac{1}{2}\cosh 2r\,(\lambda_+{}^2\,e^{\mp 2r_+} + \lambda_-{}^2\,e^{\mp 2r_-}) \mp \lambda_+\lambda_-\sinh 2r\exp(\mp 2r_s)]\,, \tag{D.7a}$$

$$\langle(\Delta\alpha_{\,\substack{12\\22}})^2\rangle = \tfrac{1}{2}[\tfrac{1}{2}\cosh 2r\,(\lambda_+{}^2\,e^{\pm 2r_+} + \lambda_-{}^2\,e^{\pm 2r_-}) \mp \lambda_+\lambda_-\sinh 2r\exp(\pm 2r_s)]\,, \tag{D.7b}$$

$$r_{\substack{s\\a}} \equiv \tfrac{1}{2}(r_+ \pm r_-)\,. \tag{D.7c}$$

The variance $\langle(\Delta\alpha_{11})^2\rangle$ is minimized, for a given r and r_+ (or r_-), when

$$2r_a \equiv r_+ - r_- = \ln(\lambda_+/\lambda_-) = (\varepsilon/\Omega)[1 + \tfrac{1}{3}(\varepsilon/\Omega)^2 + \tfrac{1}{5}(\varepsilon/\Omega)^4 + \cdots]\,. \tag{D.8}$$

When this condition is satisfied, the variance of α_{11} is

$$\langle(\Delta\alpha_{11})^2\rangle = \tfrac{1}{2}\lambda_+\lambda_-\exp(-2r_s)\exp(-2r)\,, \tag{D.9}$$

which can be arbitrarily small.

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This reference list is regrettably limited, owing to the large volume of literature on this subject. The author apologizes to the many authors whose contributions appear only indirectly, as references within these references, or not at all.

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