1 Useful Formulae, All in One Place

Fourier transformation in 3 dimensions:

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{+i\vec{p}\cdot\vec{x}}\phi(\vec{p})\,, \tag{1}$$

$$\phi(\vec{p}) = \int d^3\vec{x}\; e^{-i\vec{p}\cdot\vec{x}}\,\phi(\vec{x})\,. \tag{2}$$

Performing integrals over space:

$$\int d^3\vec{x}\; e^{i(\vec{p}+\vec{q})\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{p}+\vec{q})\,, \tag{3}$$

which forces $\vec{q} = -\vec{p}$. Similarly,

$$\int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i(\vec{x}+\vec{y})\cdot\vec{p}} = \delta^3(\vec{x}+\vec{y})\,, \tag{4}$$

forcing $\vec{x} = -\vec{y}$.

Fourier transformation in 4 dimensions: recall that

$$x \cdot p \equiv \eta_{\mu\nu}x^\mu p^\nu = -x^0p^0 + \vec{x}\cdot\vec{p}, \tag{5}$$

so

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\,\phi(p)\,, \tag{6}$$

$$\phi(p) = \int d^4x\; e^{-ip\cdot x}\,\phi(x)\,, \tag{7}$$

which look just like the 3D expressions. (Don't you like this metric?)

Lorentz invariant phase space for a particle of mass m:

$$\int \frac{d^4p}{(2\pi)^4}\, 2\pi\,\delta(p^2+m^2)\,\theta(p^0) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}}\,, \qquad E_{\vec{p}} \equiv \sqrt{\vec{p}^{\,2}+m^2}\,. \tag{8}$$

Here $E_{\vec{p}}$ is also written $\omega_{\vec{p}}$ or p^0.

Free field theory:

$$S = \int d^4x\mathcal{L}\Big[\phi(x), \partial_\mu\phi(x)\Big]\,, \tag{9}$$

$$\mathcal{L}_{\text{Klein-Gordon}} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\,, \tag{10}$$

$$\pi^\mu \equiv -\frac{\delta\mathcal{L}\big[\phi,\partial_\mu\phi\big]}{\delta\partial_\mu\phi} \quad \to_{\text{K-G}} \quad \partial^\mu\phi\,, \tag{11}$$

$$\text{Euler-Lagrange}: \quad \frac{\delta\mathcal{L}\big[\phi,\partial_\mu\phi\big]}{\delta\phi} = -\partial_\mu\pi^\mu \quad \to_{\text{K-G}} \quad \partial_\mu\partial^\mu\phi = m^2\phi\,. \tag{12}$$

The field can be expanded in creation an annihilation operators:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}} \left[a_{\vec{p}}\, e^{+ip\cdot x} + a^\dagger_{\vec{p}} e^{-ip\cdot x}\right] , \tag{13}$$

$$\left[a_{\vec{p}}\, ,\, a^\dagger_{\vec{q}}\right] = 2E_{\vec{p}}\, (2\pi)^3 \delta^3(\vec{p}-\vec{q})\, . \tag{14}$$

Note that x and the phases here are 4-vector and 4-vector product, with $p^0 \equiv E_{\vec{p}} = \sqrt{\vec{p}^2+m^2}$. The expansion ONLY works for free fields.

Two-point correlation functions:

$$G^>(x-y) \equiv \langle 0|\, \phi(x)\phi(y)\, |0\rangle\, , \tag{15}$$

$$G^<(x-y) \equiv \langle 0|\, \phi(y)\phi(x)\, |0\rangle\, , \tag{16}$$

$$\begin{aligned} G_R(x-y) &\equiv \langle 0| \left[\phi(x)\, ,\, \phi(y)\right] |0\rangle \Theta(x^0-y^0) \\ &= \left(G^>(x-y) - G^<(x-y)\right) \Theta(x^0-y^0)\, , \end{aligned} \tag{17}$$

$$\begin{aligned} G_T(x-y) &\equiv \langle 0|\, \phi(x)\phi(y)\, |0\rangle \Theta(x^0-y^0) + \langle 0|\, \phi(y)\phi(x)\, |0\rangle \Theta(y^0-x^0) \\ &= G^>(x-y)\Theta(x^0-y^0) + G^<(x-y)\Theta(y^0-x^0) \\ &= \langle 0|\, \mathbf{T}\Big(\phi(x)\phi(y)\Big)\, |0\rangle = -i\Delta(x-y)\, . \end{aligned} \tag{18}$$

Free theory values:

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} G(p) \tag{19}$$

with

$$G^>(p) = 2\pi\delta(p^2+m^2)\Theta(p^0) \tag{20}$$

$$G^<(p) = 2\pi\delta(p^2+m^2)\Theta(-p^0) \tag{21}$$

$$G_R(p) = \frac{-i}{p^2+m^2-i\epsilon\,\mathrm{sign}(p^0)} \tag{22}$$

$$G_T(p) = \frac{-i}{p^2+m^2-i\epsilon} \quad \text{or} \quad \Delta(p) = \frac{1}{p^2+m^2-i\epsilon}\, . \tag{23}$$

G_T or Δ are called the time ordered correlator or the Feynman propagator or just the propagator.

Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{24}$$

Clifford algebra:

$$\left\{\gamma^\mu\,,\,\gamma^\nu\right\} = -2g^{\mu\nu} \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{24}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \qquad (\epsilon^{0123} = -\epsilon_{0123} = 1) \tag{25}$$

Dirac equation:

$$(-i\partial\!\!\!/ + m)\psi = 0\,, \quad (p\!\!\!/ + m)u(p) = 0\,, \quad (-p\!\!\!/ + m)v(p) = 0 \tag{26}$$

Lorentz transforms

$$D(\omega) = \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\,, \qquad S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right]. \tag{27}$$

Spin contractions

$$\bar{u}_{p,s}u_{p,s'} = 2m\delta_{ss'} \quad \bar{v}_{p,s}v_{p,s'} = -2m\delta_{ss'}\,, \quad \bar{u}_{p,s}\gamma^\mu u_{p,s'} = 2p^\mu\delta_{ss'} \quad \bar{v}_{p,s}\gamma^\mu v_{p,s'} = 2p^\mu\delta_{ss'} \tag{28}$$

Spin sums

$$\sum_s u_{s,p}\bar{u}_{s,p} = -p\!\!\!/ + m\,, \qquad \sum_s v_{s,p}\bar{s}_{s,p} = -p\!\!\!/ - m\,. \tag{29}$$

Fermionic propagator

$$S(p) = \int d^4x e^{-ip\cdot x} i\langle \mathrm{T}(\psi_a(x)\bar{\psi}(0))\rangle = \frac{1}{p\!\!\!/ + m} = \frac{-p\!\!\!/ + m}{p^2 + m^2 - i\epsilon} \tag{30}$$