Physics 610 Homework 1

Due Wed. 19 September 2012

1 4-vector notation and Maxwell equations

The purpose of this problem is to get you used to index notation and in particular to 4-vector notation.

Recall that the electric and magnetic fields can be derived in terms of two quantities, the scalar potential Φ and the vector-potential $\vec{A}$. It is also convenient to replace the magnetic field with the antisymmetric 2-tensor F_{ij} defined as[1]

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i \qquad \text{(non-covariant index notation).} \tag{1}$$

In terms of this, $F_{12} = B_3$, $F_{23} = B_1$, and $F_{31} = B_2$, *e.g.*,

$$F_{ij} = \epsilon_{ijk}\, B_k \qquad \text{and} \qquad B_i = \frac{\epsilon_{ijk}}{2} F_{jk}\,. \tag{2}$$

Here as usual ϵ_{ijk} is the totally antisymmetric symbol (Levi-Civita tensor) with $\epsilon_{123} = 1$.

In terms of ∂_t, ∂_i, A_i, and Φ, write the standard (non-covariant) expressions for the electric and magnetic fields E_i and F_{ij}.

Now we move to 4-vector notation. Define $A^\mu = (\Phi, \vec{A})$ (where $\mu = 0, 1, 2, 3$ and the notation means that for $\mu = 0$ you choose the first object in the parenthesis and for $\mu = 1, 2, 3$ you choose the component of the second, *e.g.*, $A^0 = \Phi$ and $A^{1,2,3} = \vec{A}_{1,2,3}$. Careful: while $A^0 = \Phi$, we have $A_0 = -\Phi$.) Also define $\partial_\mu = (\partial_t, \partial_i)$. Introduce

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{3}$$

and show how F^{0i} is related to the electric field and how F^{ij} is related to the magnetic tensor. Here F^{0i} means $F^{\mu\nu}$ for the case where $\mu = 0$ but $\nu \neq 0$–we will use Roman letters to mean that a Lorentz index μ is not zero.

Also introduce $j^\mu = (\rho, \vec{j})$ the 4-current. Show that the covariant equation

$$\partial_\mu F^{\nu\mu} = [\pm] j^\nu \tag{4}$$

is equivalent to both Gauss' law and Ampere's law. Figure out which is the correct sign on the current; is my $\pm$ a $+$ or a $-$?

[1] As in class, Roman indices refer to the three space indices 1,2,3 corresponding to x, y, z; since $g_{ij} = \delta_{ij}$, we make no distinction between upper and lower indices. Greek indices are 4-vector indices, and it is essential to distinguish between upper and lower indices.

Now define $\epsilon_{\mu\nu\alpha\beta}$ the 4D antisymmetric symbol which generalizes the 3-D Levi-Civita tensor: $\epsilon_{\nu\mu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}$ and similarly for any other permutation of the indices, and $\epsilon_{0123} = -1$. [The minus sign is so that $\epsilon^{0123} = +1$; the sign flips because an odd number of the $g^{\mu\nu}$'s you need to raise the indices are negative.]

Show that

$$\epsilon_{\mu\nu\alpha\beta}\partial^{\nu} F^{\alpha\beta} = 0 \tag{5}$$

is an identity (is true regardless of what values A^{μ} take provided they are twice differentiable) and that this identity is equivalent both to Gauss' Law for magnetism and to Faraday's law.

You are now an expert with index notation.

2 Condition to be a Lorentz transformation

Here we clear up two simple pieces of the derivation of what is and is not a Lorentz transformation.

In class we saw that $\Lambda^{\mu}{}_{\nu}$ is a Lorentz transformation if and only if

$$x^{\alpha}\Lambda^{\mu}{}_{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} x^{\beta} = x^{\alpha} g_{\alpha\beta} x^{\beta} \tag{6}$$

for *any* choice of 4-coordinate x^{α}. Show that this really does require that

$$\Lambda^{\mu}{}_{\alpha} g_{\mu\nu} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta} \tag{7}$$

should hold. Hint: show that if (7) is NOT true, then there is some x^{μ} such that (6) is also NOT true. Then argue by contrapositive.

Next, consider

$$\Lambda^{\mu}{}_{\nu} = \exp \omega^{\mu}{}_{\nu} \tag{8}$$

where multiplication is defined by thinking of the first upper index as a column index and the second lower index as a row index, and using matrix multiplication, *eg*,

$$\exp \omega^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} + \frac{1}{2}\omega^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu} + \frac{1}{6}\omega^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\beta}\omega^{\beta}{}_{\nu} + \frac{1}{24}\ldots \tag{9}$$

Show that, provided $\omega_{\mu\nu} = -\omega_{\nu\mu}$, that $\Lambda^{\mu}{}_{\nu}$ really is a Lorentz transform, that is, that it satisfies Eq. (7).

3 Translations and Lorentz Transformations

Let's see that translations and Lorentz transformations do not, in general, commute. Consider the translation which shifts the coordinates by a distance ξ^{μ}:

$$x^{\mu}_{\mathrm{tr}} = x^{\mu} + \xi^{\mu} \tag{10}$$

and the Lorentz transformation which rotates the 4-coordinates according to

$$x^\mu_{\rm lt} = \Lambda^\mu{}_\nu x^\nu = x^\mu + \omega^\mu{}_\nu x^\nu \tag{11}$$

(working to linear order in small ω!)

Show that, if we *first* apply the Lorentz transform on the coordinates and *then* perform the tranlation, the coordinate will change to

$$x^\mu_{\rm lthent} = x^\mu + \omega^\mu{}_\nu x^\nu + \xi^\mu \tag{12}$$

whereas, if we *first* apply the translation and *then* apply the Lorentz transformation, we get

$$x^\mu_{\rm tthenl} = x^\mu + \omega^\mu{}_\nu x^\nu + \xi^\mu + \omega^\mu{}_\nu \xi^\nu \tag{13}$$

which is not the same. Why can the difference be interpreted as a translation?

At lowest order in ξ, the unitary operator which implements the translation is

$$U(\xi) = \mathbf{1} - i\hat{P}_\mu \xi^\mu \,, \tag{14}$$

and the unitary operator which performs the Lorentz transformation is

$$U(\omega) = \mathbf{1} + \frac{i}{2}\omega_{\mu\nu}\hat{M}^{\mu\nu} \,. \tag{15}$$

Argue that $U(\omega)U(\xi)$ should differ from $U(\xi)U(\omega)$ by $-i\hat{P}_\mu \omega^\mu{}_\nu \xi^\nu$ in order to account for the difference between Eq. (12) and Eq. (13). Show that the difference between $U(\omega)U(\xi)$ and $U(\xi)U(\omega)$ arises due to the failure of M and P to commute as operators. Then show that, to give the right difference for *arbitrary* (small) $\omega_{\mu\nu}$ and ξ^α, the operators M and P must obey the commutation relations

$$\Big[P^\mu \,,\, M^{\nu\alpha}\Big] = i\Big(g^{\mu\alpha}P^\nu - g^{\mu\nu}P^\alpha\Big) \,. \tag{16}$$

4 M, J, K, N, $N^\dagger$

Define $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ and $K_i = M^{i0}$.

First, write explicitly what each J_1, J_2, J_3 are in terms of M^{ij}'s. For instance, show that $J_1 = M^{23}$.

Next, show that the commutation relations for $M^{\mu\nu}$,

$$\Big[M^{\mu\nu} \,,\, M^{\alpha\beta}\Big] = i\Big(g^{\mu\alpha}M^{\nu\beta} + g^{\nu\beta}M^{\mu\alpha} - g^{\mu\beta}M^{\nu\alpha} - g^{\nu\alpha}M^{\mu\beta}\Big) \tag{17}$$

turn into the commutation relations

$$\begin{aligned}
\Big[J_i\,,\,J_j\Big] &= i\epsilon_{ijk}J_k\,,\\
\Big[J_i\,,\,K_j\Big] &= i\epsilon_{ijk}K_k\,,\\
\Big[K_i\,,\,K_j\Big] &= -i\epsilon_{ijk}J_k\,.
\end{aligned} \tag{18}$$

Next, introduce

$$N_i \equiv \frac{J_i - iK_i}{2} \tag{19}$$

and its Hermitian conjugate

$$N_i^\dagger \equiv \frac{J_i + iK_i}{2} \tag{20}$$

(recall that the $M^{\mu\nu}$ are Hermitian operators, so $N^\dagger$ really is the Hermitian conjugate of N). Show, using Eq. (18), that N, $N^\dagger$ satisfy the commutation relations

$$\begin{aligned}
\Big[N_i\,,\,N_j\Big] &= i\epsilon_{ijk}N_k\,,\\
\Big[N_i^\dagger\,,\,N_j^\dagger\Big] &= i\epsilon_{ijk}N_k^\dagger\,,\\
\Big[N_i\,,\,N_j^\dagger\Big] &= 0\,.
\end{aligned} \tag{21}$$

5 Extra credit

Verify from the defining condition for $\Lambda^\mu{}_\nu$,

$$g_{\mu\nu} = \Lambda^\alpha{}_\mu g_{\alpha\beta}\Lambda^\beta{}_\nu \qquad \text{or} \qquad g = \Lambda^\top g\Lambda\,, \tag{22}$$

that the group $\mathcal{O}(3,1)$ of all Λ satisfying this property, together with the product rule

$$\Lambda^\mu_{12\nu} = \Lambda^\mu_{1\,\alpha}\Lambda^\alpha_{2\,\nu}\,, \tag{23}$$

constitute an abstract *group*. That is, show that they obey the four criteria

1. Closure (the product of two elements of the group is also an element of the group)
2. Associativity
3. Existence of an identity
4. Existence of inverses