Physics 610   Homework 10 and Final

Due Thursday 13 December.

This homework is extra credit only. There are two problems. Do one, both, or neither, as you have time available.

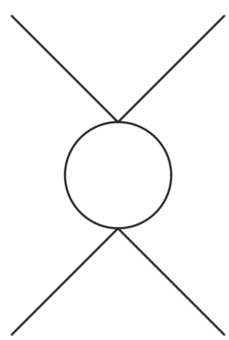
1 Renormalization

Consider scalar ϕ^4 theory, with one real scalar field and Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda_0}{24}\phi^4 \,. \tag{1}$$

We have seen many times that the lowest-order matrix element for the scattering process $\phi\phi \to \phi\phi$ is $\mathcal{M} = \lambda_0$. Here we consider the next order.

Label the two incoming momenta p, k and the outgoing momenta p', k'. Consider the diagram



where p enters and p' exits the upper vertex and k enters and k' exits the lower vertex. (There are two similar diagrams with different external momentum labels, we will figure them out based on what happens with this diagram.)

1.1 The evaluation

Show that the contribution to the matrix element from this diagram is

$$i\frac{(-i\lambda_0)^2}{2}\int \frac{d^D l}{(2\pi)^D}\frac{(-i)^2}{(l^2+m^2-i\epsilon)((l+p-p')^2+m^2-i\epsilon)} \tag{2}$$

and label l and $l+p-p'$ on a picture of the diagram.

Combine the denominators using the Feynman parameter trick. Then Wick rotate the l^0 variable. Write an expression for the diagram in terms of your $\int dx$ Feynman integral, $\int d^D l_E$,

m^2, and Mandelstamm t. Now set $m^2 = 0$ from now through the rest of the calculation, to make the problem a little simpler.

Perform the l-integration in dimensional regularization. Then perform the x integration by using $\int_0^1 dx \ln[x(1-x)] = -2$. You should find (using $D = 4 - 2\epsilon$, your $1/\epsilon$ will differ if you use the convention of the book)

$$-\frac{\lambda_0^2}{32\pi^2}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-t} + 2\right). \tag{3}$$

Add the other two contributions to find

$$\mathcal{M} = \lambda_0 - \frac{\lambda_0^2}{32\pi^2}\left(\frac{3}{\epsilon} + 6 + \ln\frac{\mu^6}{|stu|} - i\pi\right) \tag{4}$$

(Don't worry about showing that the imaginary part is correct, just assume that it arises from the difference between $\ln(-s)$ and $\ln(s)$.)

1.2 The cross-section

Compute the total scattering cross-section which you obtain using the lowest order result $\mathcal{M} = \lambda_0$. Make the strict $m^2 = 0$ approximation to simplify the calculation. Your answer should be a function of λ_0 and s only. [Hint: The whole phase space integral reduces to the somewhat simpler expression

$$\sigma = \frac{1}{16\pi s^2}\int_{-s}^{0} dt \tag{5}$$

see, eg, Burgess and Moore, page 196, Eq. (6.22) and (6.25). Remember that the identical final states mean that you should integrate over half the t-range, or include an extra factor of $\frac{1}{2}$ to avoid double counting the available final states.]

Now re-compute the scattering cross-section to order λ_0^3 by including the corrections you found to $\mathcal{M}$. Now your result should be a function of s, μ^2/s, and $1/\epsilon$ as well.

1.3 Renormalization

Supposing we *define* $\lambda_r(\mu_1)$ so that the cross-section for $s = \mu_1^2$ equals the lowest-order cross-section evaluated using λ_r; that is, equate the order-λ_0^3 cross-section you found in the last subsection with the leading-order expression but using λ_r.

Write this relation, which expresses $\lambda_r(\mu_1)$ in terms of λ_0. Now, invert this relation to find an expression for λ_0 in terms of λ_r, accurate to second order. (Hint: if $x = y + y^2$ and both x and y are small, then the y^2 can be approximated as x^2, so $y = x - x^2$ to the desired order of accuracy.)

Substitute this relationship into your previous result for $\mathcal{M}$ (at general s, t, u), working to second order in $\lambda_r(\mu_1)$. You should find that all reference to $1/\epsilon$ disappears. However your answer should depend on μ_1^6/stu.

You have now successfully renormalized a theory to one loop order.

1.4 Scale dependence of the coupling

As a final exersize, suppose that you use $\lambda_r(\mu_1)$ but your friend chooses a different value and uses $\lambda_r(\mu_2)$. Use the expressions from the last section to relate each coupling to λ_0, and then use the two relations to eliminate λ_0 and directly relate $\lambda_r(\mu_1)$ to $\lambda_r(\mu_2)$. You should find an expression of form

$$\lambda_r(\mu_2) = \lambda_r(\mu_1) + \lambda_r^2(\mu_1) \times (\text{something}) \times \ln \frac{\mu_2^2}{\mu_1^2}\,. \tag{6}$$

If $\mu_2 > \mu_1$, which coupling is larger?

2 an elementary QED calculation

Consider the theory QED of electrons and muons. There is a gauge field A^μ representing electromagnetism, as well as *two* Dirac fermion fields ψ_1 and ψ_2, with masses m_1 and m_2. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_2(i\not{D} - m_2)\psi_2 + \bar{\psi}_1(i\not{D} - m_1)\psi_1\,. \tag{7}$$

Here $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$.

2.1 Scattering

Using the book to find the Feynman rules if you prefer, draw the single diagram responsible for the scattering process $\psi_1\psi_2 \to \psi_1\psi_2$. Label the ψ_1 and ψ_2 incoming momenta p, k and the ψ_1 and ψ_2 outgoing momenta p' and k' respectively. Write down the associated scattering matrix element. Find its Hermitian conjugate. Write the product of the two, summed over final and averaged over initial spins. Now take the limit $m_1, m_2 \to 0$. Then carry out the traces and find an expression for the squared matrix element. You should find

$$|\overline{\mathcal{M}}|^2 = 2e^4 \frac{s^2 + u^2}{t^2}\,. \tag{8}$$

2.2 Annihilation and crossing

Now consider the process in which ψ_1 and its antiparticle $\bar{\psi}_1$ annihilate to give ψ_2 and its antiparticle $\bar{\psi}_2$. Call the ψ_1 incoming momentum p and the $\bar{\psi}_1$ momentum k, while the outgoing ψ_2 and $\bar{\psi}_2$ momenta are p' and k'.

Draw the single diagram contributing to this process. Write out the matrix element and the spin summed-and-averaged squared matrix element – but do *not* evaluate the traces and contract.

Look at your two expressions for spin averaged squared matrix elements – for the $\psi_1\psi_2 \to \psi_1\psi_2$ process and the $\psi_1\bar{\psi}_1 \to \psi_2\bar{\psi}_2$ process. Argue that they are identical, except that the four momenta $(p, k, -p', -k')$ in the first expression have been switched to $p, -k', k, -p'$ in the second expression.

Now write the expressions for s, t, u (setting all masses to zero). Apply the same permutation – what does each Mandelstamm variable turn into?

Use this information to show, *without* performing any Dirac traces or computing any farther, that the spin summed-averaged matrix element squared for the annihilation process is

$$|\overline{\mathcal{M}}|^2 = 2e^4 \frac{t^2 + u^2}{s^2} \,. \tag{9}$$

The property you have just used (that changing particles between the initial and final states corresponds to changing the choice and sign of their momenta and leads to a re-assignment of Mandelstamm variables) is called *crossing symmetry*.