Physics 610 Homework 2

Due Thurs 27 September 2012

1 Commutation relations

In class we found that

$$\Big[\phi(\mathbf{x})\,,\,\pi(\mathbf{y})\Big] = i\delta^3(\mathbf{x}-\mathbf{y}) \tag{1}$$

and then defined

$$\tilde{\phi}(p_m) \equiv L^{-3/2}\int d^3\mathbf{x} e^{-i\mathbf{p}_m\cdot\mathbf{x}}\phi(\mathbf{x}) \tag{2}$$

and likewise for $\tilde{\pi}(p_m)$. Show that it really follows from these definitions that

$$\Big[\tilde{\phi}(p_n)\,,\,\tilde{\pi}(p_m)\Big] = i\delta_{p_n,-p_m}\,. \tag{3}$$

Then, show that

$$\frac{1}{2}\int d^3x(\nabla\phi(x))^2 = \frac{1}{2}\sum_{p_n} p_n^2\tilde{\phi}(p_n)\tilde{\phi}(-p_n)\,. \tag{4}$$

Next, show that the definition

$$a_{p_n} = \frac{\omega_p\tilde{\phi}(p_n) + i\pi(p_n)}{\sqrt{2\omega_p}} \qquad \text{leads to} \qquad a^\dagger_{p_n} = \frac{\omega_p\tilde{\phi}(-p_n) - i\pi(-p_n)}{\sqrt{2\omega_p}} \tag{5}$$

which together lead to

$$\Big[a_{p_n}\,,\,a^\dagger_{p_m}\Big] = 1\delta_{p_n,p_m}\,. \tag{6}$$

Finally, fill in the steps to show that

$$H = \frac{1}{2}\sum_{p_n}\tilde{\pi}(p_n)\tilde{\pi}(-p_n) + (p^2+m^2)\tilde{\phi}(p_n)\tilde{\phi}(-p_n) = \frac{1}{2}\sum_{p_n}\omega_p(a^\dagger_{p_n}a_{p_n} + a_{p_n}a^\dagger_{p_n})\,. \tag{7}$$

(That is, take the first expression for H to be shown, and derive the second expression from it, together with the definition of a and $a^\dagger$.)

2 Momentum and field

This is part of problem 3.4 from the book. Recall that the translation operator is $T(a) \equiv \exp(-iP^\mu a_\mu)$ and its action on the field is

$$T(a)^{-1}\phi(x)T(a) = \phi(x-a)\,. \tag{8}$$

By considering infinitesimal a and the Taylor expansion of $\phi(x-a)$ about $\phi(x)$, find the expression for $\Big[P^\mu\,,\,\phi(x)\Big]$. Show that the time component ($\mu=0$) of your relation is the Heisenberg equation of motion (recall $H=P^0$)

$$\partial_t\phi = i\Big[H\,,\,\phi\Big]\,. \tag{9}$$

3 Correlation functions

Here we will study the values of several correlation functions in the free theory. First, use the expansion of the field in terms of creation and annihilation operators and their commutation relations (Useful Formulae equations 13,14)

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3 2k^0}\left(a_k e^{ik\cdot x} + a_k^\dagger e^{-ik\cdot x}\right)_{k^0\equiv\sqrt{\vec{k}^2+m^2}} \\
&= \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2+m^2)\Theta(k^0)\left(a_k e^{ik\cdot x} + a_k^\dagger e^{-ik\cdot x}\right) && (10)\\
\Big[a_k\,,\,a_p^\dagger\Big] &= 2k^0(2\pi)^3\delta^3(\vec{k}-\vec{p}) && (11)
\end{aligned}$$

to derive that the Wightman correlation function

$$G^>(x) \equiv \langle 0|\,\phi(x)\phi(0)\,|0\rangle \tag{12}$$

takes the value

$$G^>(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\, 2\pi\delta(k^2+m^2)\Theta(k^0)\,. \tag{13}$$

Hint: insert the explicit expressions for $\phi(x)$ and $\phi(0)$ in terms of creation and annihilation operators. Use that $a\,|0\rangle = 0$ and $\langle 0|\,a^\dagger = 0$ and the commutation relation to evaluate the resulting expression.

Next, use translation invariance to show that the "lesser than" Wightman correlator

$$G^<(x) \equiv \langle 0|\,\phi(0)\phi(x)\,|0\rangle = G^>(-x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\, 2\pi\delta(k^2+m^2)\Theta(-k^0)\,. \tag{14}$$

It is also common to define the retarded Green function

$$G_R(x) \equiv \langle 0|\,\Big[\phi(x)\,,\,\phi(0)\Big]\,|0\rangle\;\Theta(x^0) = \Big(G^>(x) - G^<(x)\Big)\Theta(x^0) \tag{15}$$

the advanced Green function

$$G_A(x) \equiv G_R(-x) = -\,\langle 0|\,\Big[\phi(x)\,,\,\phi(0)\Big]\,|0\rangle\;\Theta(-x^0) \tag{16}$$

and the time-ordered Green function

$$G_T(x) \equiv G^>(x)\Theta(x^0) + G^<(x)\Theta(-x^0)\,. \tag{17}$$

Explain why $G^{>}(x) = G^{<}(x)$ for spacelike x, and how that ensures that G_{R} vanishes unless $x^0 \geq |\vec{x}|$, G_{A} vanishes unless $x^0 \leq -|\vec{x}|$, and all three functions are Lorentz invariant (despite the presence of the non-Lorentz-invariant $\Theta(x^0)$ in their expressions).

Although we know the momentum-space form of $G^{>}(x)$ and $G^{<}(x)$, the presence of $\Theta(x^0)$ in the above expressions makes it tricky to get to the momentum-space expressions for these correlation functions. But we can do so by writing

$$\Theta(x^0) = \lim_{\epsilon \to 0^+} \int \frac{d\omega}{2\pi i} \, \frac{e^{i\omega x^0}}{\omega - i\epsilon} \,. \tag{18}$$

Substitute this expression, as well as the Fourier-space expressions Eq. (13) and Eq. (14), into Eq. (15). Use the $\delta(k^2 + m^2)$ functions to perform the k^0 integration, to show that

$$G_{R}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int \frac{d\omega}{2\pi i} \, \frac{e^{i\omega x^0}}{\omega - i\epsilon} \left(\frac{e^{-i\omega_k x^0}}{2\omega_k} - \frac{e^{i\omega_k x^0}}{2\omega_k} \right) \qquad \text{where } \omega_k = \sqrt{\vec{k}^2 + m^2} \,. \tag{19}$$

This expression is the sum of two terms with different-appearing complex exponents. For each term, change variables from ω to $k^0 \equiv -\omega \pm \omega_k$ such that the new variable appears in the exponent as $e^{-ik^0 x^0}$. Re-organize the integrand in terms of this variable and show that it gives

$$G_{R}(x) = \int \frac{d^3\vec{k} dk^0}{(2\pi)^4} \frac{-ie^{ik\cdot x}}{\vec{k}^2 + m^2 - (k^0 + i\epsilon)^2} \,. \tag{20}$$

Therefore the momentum-space value of G_{R} is $G_{R}(k) = -i/(k^2 + m^2)_{k^0 \to k^0 + i\epsilon}$.

You will take my word that one can similarly show that $G_{T}(k) = -i/(k^2 + m^2 - 2i\epsilon\omega_k)$.

One last thing. The retarded function we found has poles in the complex k^0 plane at the points $k^0 = -i\epsilon \pm \omega_k$, but it is analytic in the upper half-plane (whenever k^0 has positive imaginary part). Show that this is actually totally general. That is, defining the Fourier transform of the retarded function

$$G_{R}(k) = \int d^4x e^{-ik_\mu x^\mu} G_{R}(x) \,, \tag{21}$$

argue that so long as $G^{>}(x)$ and $G^{<}(x)$ are *nonsingular* for $x^0 > 0$, that the integral is perfectly well defined if k^0 has a positive imaginary part. (Hint: write out the exponent $e^{-ik_\mu x^\mu} = e^{ik^0 x^0} e^{-i\vec{k}\cdot\vec{x}}$, and then write $k^0 = k^0_{\rm re} + ik^0_{\rm im}$. Does the exponent help or hurt convergence when $x^0 > 0$? Does it help or hurt when $x^0 < 0$? But what is the value of $G_{R}(x)$ when $x^0 < 0$?) Similarly, $G_{A}(p)$ is well defined for negative imaginary part.

4 Energy associated with the vacuum

We saw that the Hamiltonian for a free field theory is

$$H = \sum_{\vec{p} = \frac{2\pi}{L}\vec{n}} \frac{\omega_p}{2} \left(a^\dagger_{p_n} a_{p_n} + a_{p_n} a^\dagger_{p_n} \right) \tag{22}$$

where $\left[a_{p_n}\,,\, a^\dagger_{p_m}\right] = \delta_{\vec{n},\vec{m}}$ (the Kroneker delta).

Show that the energy of the vacuum $\langle 0|\, H\, |0\rangle$ is

$$E_{\rm vac} \equiv \langle 0|\, H\, |0\rangle = \sum_{\vec{p}} \frac{\omega_p}{2}\,. \tag{23}$$

Suppose that, because space is somehow discrete on some scale a, there is a maximum size which the momentum p is allowed to take, call it $p_{\rm max}$.

Show that the energy of the vacuum is extensive – that is, that it grows as we increase the size of our "box" as L^3. This is to be expected. Then show that it also depends on $p_{\rm max}$ with proportionality $p^4_{\rm max}$, that is, $E_{\rm vac} \sim L^3 p^4_{\rm max}$. Do not try to evaluate the actual coefficient, which depends on exactly what we mean by "momenta cannot exceed some scale $p_{\rm max}$."

Show that, if we include a constant term C in the Lagrangian density, the Hamiltonian is shifted by $-L^3 C$.

[This energy associated with space is of no relevance in particle physics because it is not observable. Particle physicists who also worry about gravity *do* worry about it, though, because it should behave as a "cosmological constant," and we know observationally that the cosmological constant is very small, whereas presumably $p^4_{\rm max}$ is very large. In principle it might be that C is just the right value to balance the energy associated with all the SHO zero-points, but . . . .]