Physics 610 Homework 3

Due Thurs 4 October 2012

1 Projection operators

Consider the free field theory of one scalar ϕ of mass m. Define the state

$$|p\rangle = a_p^\dagger |0\rangle . \tag{1}$$

(Recall that $\left[a_k \, , \, a_p^\dagger\right] = 2\omega_p (2\pi)^3 \delta^3(\vec{k} - \vec{p})$ and that $a_k |0\rangle = 0$.) Explain that the object

$$\hat{F}_{\text{range}} \equiv \int_{p \in \text{range}} \frac{d^3p}{(2\pi)^3 2\omega_p} |p\rangle \langle p| \tag{2}$$

is an operator. (Here ($p \in$ range) means that the integral is restricted to some well specified range of momenta p. For instance, the range could be $|\vec{p}| < q$ for some q, or $p^z > q_1$ but $q_2 < p^x < q_3$ and p^y anything, or ... Each choice for the range included in the integral gives rise to a distinct operator; we want to make general statements about such operators.)

An operator $\hat{F}$ is said to be a projection operator if it satisfies the property $(\hat{F})^2 = \hat{F}$. Show that $\hat{F}_{\text{range}}$, defined above, is a projection operator.

2 Time ordering symbol

The time ordering symbol T means that the operators (inside parenthesis or appearing to its right) are to be re-arranged in descending order of their time arguments. For two fields this means

$$\mathrm{T}(\phi(x)\phi(y)) \equiv \begin{cases} \phi(x)\phi(y) \, , & x^0 > y^0 \\ \phi(y)\phi(x) \, , & y^0 > x^0 \, . \end{cases} \tag{3}$$

We can rewrite this condition using the Heaviside function $\Theta(r)$ which is 1 if $r > 0$ and 0 if $r < 0$, specifically,

$$\mathrm{T}(\phi(x)\phi(y)) = \phi(x)\phi(y)\Theta(x^0 - y^0) + \phi(y)\phi(x)\Theta(y^0 - x^0) \, . \tag{4}$$

First, check that the two expressions are the same by seeing that they give the same result for each possible case $x^0 > y^0$ or $y^0 > x^0$. (This should be trivially easy.) Next, consider the case with three operators:

$$\mathrm{T}(\phi(x)\phi(y)\phi(z)) \equiv \begin{cases} \phi(x)\phi(y)\phi(z) \, , & x^0 > y^0 > z^0 \\ \phi(x)\phi(z)\phi(y) \, , & x^0 > z^0 > y^0 \\ \phi(y)\phi(x)\phi(z) \, , & y^0 > x^0 > z^0 \\ \cdots & \cdots \end{cases} \tag{5}$$

Fill in the remaining terms. Then write an expression in terms of Θ functions for this time-ordered product.

If we considered the time ordered product of N fields, how many terms would appear in its explicit expression? (This is the reason we use the time ordering operator!)

3 Flux factor

In lecture, in discussing the longitudinal part of the wave packets for scattering, we reached the expression

$$\int \frac{dp'_{1z} dp''_{1z} dp'_{2z} dp''_{2z}}{(2\pi)^4 (2E_1 2E_2)^2} \; \psi^*(p''_{1z}) \psi(p'_{1z}) \psi^*(p''_{2z}) \psi(p'_{2z}) \\ \times (2\pi)^2 \delta(p'_{1z} + p'_{2z} - p''_{1z} - p''_{2z}) \delta(p_1^{0\prime} + p_2^{0\prime} - p_1^{0\prime\prime} + p_2^{0\prime\prime}) \tag{6}$$

with $E \equiv \sqrt{p^2 + m^2}$, the p, m arguments of each E are implicit, and

$$\int \frac{dp_z}{(2\pi) 2E} \; \psi^*(p_z) \psi(p_z) = 1 \,. \tag{7}$$

I then claimed that (for wave packets tightly peaked in momentum) the integrals could be performed and give

$$\frac{1}{2E_1 2E_2 |v_1 - v_2|} \tag{8}$$

with v_1 the group velocity along the z axis of particle 1. (Group velocity is defined in the usual way as dE/dp_z.)

Fill in the missing steps to complete this derivation. Hint: use the p_z delta function to perform the p''_{2z} integration. Remember that p^0 is a dependent variable, defined as $p^0 = \sqrt{p_z^2 + m^2}$. When you do the p''_{2z} integral, forcing $p''_{2z} = p'_{1z} + p'_{2z} - p''_{1z}$, this substitution must be made in $p^{0\,\prime\prime}_{2z}$. That means that using the remaining delta function to do the p''_{1z} integration will lead to a nontrivial Jacobian, which you have to take proper account of.

Next, show that the resulting factor, Eq. (8), is Lorentz invariant. First, show that it is unchanged by boosts along the beam axis. Then show that it equals

$$\frac{1}{2E_1 2E_2 |v_1 - v_2|} = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \,, \tag{9}$$

which is manifestly Lorentz invariant.

4 Wave packets

First, to have a clearer picture of what we are talking about when we write down a wave packet for a particle, write down a 3-dimensional wave packet $\psi(\vec{x})$ which describes a particle centered at momentum $\vec{p}$, with a wave packet centered at position $\vec{y}$, with a Gaussian distribution of momenta about $\vec{p}$ of width $\Delta \ll |\vec{p}|$. Demonstrate explicitly that it is properly normalized so that

$$\int d^3\vec{x}\psi^*\psi(\vec{x}) = 2E_{\vec{p}}\,. \tag{10}$$

Next, to have a clearer picture about the temporal part of the wave packet, consider

$$\psi(t) = e^{-i\omega' t}e^{-t^2/2\tau^2} / \left(\tau\sqrt{2\pi}\right)\,, \tag{11}$$

which is an example of the function of time we used to create a particle of energy ω'. Show that its Fourier transform $\psi(\omega)$ is peaked at $\omega = \omega'$, with peak value 1.

Consider using $\psi(t)$ to produce a single particle state for a single simple harmonic oscillator with energy splitting ω and creation operator $a^\dagger$, via the operator

$$\mathcal{O} = \int dt\; \psi(t)\, a^\dagger(t)\,. \tag{12}$$

First, write down what $a^\dagger(t)$ is in terms of $a^\dagger(t=0)$. You should be able to determine this from any quantum mechanics textbook. Then consider the state $\mathcal{O}\,|0\rangle$. Evaluate its square, and comment on its dependence on the difference $\omega' - \omega$.

Extra credit; suppose that instead of Eq. (12), the operator contains a mix of single and double particle creation operators,

$$\mathcal{O} = \int dt\; \psi(t) \left(a^\dagger(t) + [a^\dagger(t)]^2\right). \tag{13}$$

For this part, choose $\omega' = \omega$ the actual frequency of the SHO. Show that the state $\mathcal{O}\,|0\rangle$ contains a mix of the first and second excited states of the SHO, but that, if $\omega\tau \gg 1$, that the amplitude of the second excited state is exceedingly small.

5 Simplest path integral

The simplest path integral is the 0-dimensional path integral, which is just a Gaussian integral over one variable:

$$Z = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2}\,. \tag{14}$$

In analogy with what we do for the full path integral to evaluate correlation functions, generalize this to be a function of an external current J:

$$Z(J) = \int \frac{d\phi}{\sqrt{2\pi}}\; \exp\left(-\frac{\phi^2}{2} + J\phi\right)\,. \tag{15}$$

Evaluate each of the following:

$$\left.\frac{d^2 Z(J)}{dJ^2}\right|_{J=0} \qquad \text{and} \qquad \left.\frac{d^4 Z(J)}{dJ^4}\right|_{J=0} \tag{16}$$

by each of the following two methods:

1. Carry out the derivatives with respect to J. Then, set $J = 0$. Then, do the ϕ integration.

2. Solve for $Z(J)$ by actually doing the ϕ integration first. Do this by completing the square. Then, carry out the derivatives on the resulting expression, and set $J = 0$ at the end.

You should get the same answers. Verify by looking at the first method, that the answers are the two-point and four-point functions, that is, the integral with ϕ^2 and with ϕ^4 inserted into the integrand.