Physics 610 Homework 4

Due Monday 15 October 2012

1 Asymptotic series

Consider the "baby" or "toy" version of scalar ϕ^4 theory, where it is just a single integral;

$$Z = \int_{-\infty}^{\infty} d\phi \, \exp\left(\frac{-\phi^2}{2} + \frac{-\lambda\phi^4}{24} \right) . \tag{1}$$

This is what the path integral for scalar ϕ^4 theory would look like if there were only one point in spacetime (and after rotating the contour for ϕ so the i's go away).

Consider Z as a function of λ.

1.1 Values

Evaluate

- Z(0)
- Z(0.01)
- Z(0.1)
- Z(0.4)
- Z(1)
- Z(5)

numerically to 20 digits, for instance, with Maple.

1.2 Series expansion

Replace $\exp(-\lambda\phi^4/24)$ with its series expansion in λ (or equivalently, in ϕ). Find explicitly the λ^0 and λ^1 terms in the series. Evaluate the λ^0 and the sum of λ^0 and λ^1 terms (first and second partial sums) numerically, for each of the examples you did above. For which cases does the λ^1 term help improve the accuracy?

1.3 Asymptotic series

Find the complete series expansion in λ in closed form, that is, write

$$Z(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n , \tag{2}$$

and find an explicit expression for c_n. (Do this by expanding the exponent, exchanging orders of summation and integration, and doing the integral for each term in the series.)

Show that the radius of convergence (in λ) of this series is *zero*.

1.4 So what good is it?

The expansion

$$e^{-x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots \tag{3}$$

has the following property for $x > 0$: the partial sums

$$f_n(x) \equiv \sum_{m=0}^{n} \frac{(-1)^m}{m!} x^m \tag{4}$$

are alternately strict over-estimates and strict under-estimates of the actual function; that is, for $x > 0$, $f_0(x) = 1 > e^{-x}$, $f_1(x) = (1-x) < e^{-x}$, $f_2(x) = (1-x+x^2/2) > e^{-x}$, and so forth with the $<, >$ alternating. (Extra credit: prove this.)

Use this property to *prove* that the partial sums found above, Eq. (2) with n cut off at $0, 1, 2, 3, \ldots$, are alternately over-estimates and under-estimates. Therefore, the true answer always lies between neighboring terms in the series of partial sums.

Use this property to find a *bound* for $Z(\lambda)$ at $\lambda = 1$, by evaluating alternating terms until they start to diverge. How tight is the bound?

Repeat for $Z(0.4)$ and $Z(0.1)$. Argue that the bound becomes tighter and tighter as λ gets smaller, so at small λ, while the series does not converge, it gives us very good information about the value of $Z(\lambda)$.

A series with this property–zero radius of convergence but the ability to give good information near the origin–is called an *Asymptotic Series*.

1.5 Negative λ

What happens when $\lambda < 0$?

2 Cancellation of vacuum bubbles

Consider a theory of one real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\phi(\partial_\mu \partial^\mu - m^2)\phi - \frac{\lambda}{24}\phi^4 \,, \tag{5}$$

with corresponding path integral

$$\begin{aligned} Z(J) &= \int \mathcal{D}\phi \exp\left(i \int d^4u \mathcal{L}(\phi(u))\right) \\ &= \exp\left(\frac{-i\lambda}{24}\int d^4u \frac{\delta}{(i\delta J(u))^4}\right) N \exp\left(\frac{1}{2}\int d^4x d^4y \, iJ(x)\, \Delta_0(x-y)\, iJ(y)\right) \,, \end{aligned} \tag{6}$$

where N is an overall normalization factor which we will not need, and $\Delta_0(x-y) = \langle 0| \mathbf{T}\left(\phi(x)\phi(y)\right)|0\rangle_{\lambda=0}$ is the two-point time-ordered correlation function of the free theory. In this problem we will be interested in the 4-point correlation function.

2.1 Lowest order

Compute the 4-point function $\langle 0| \mathbf{T}\left(\phi(x)\phi(y)\phi(z)\phi(w)\right)|0\rangle$ at leading order in λ, which is *zero order*. Do this by explicitly evaluating

$$G(x,y,z,w) \equiv \frac{1}{Z(J)} \left.\frac{\delta^4 Z(J)}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J=0} \tag{7}$$

at zero order in λ, that is, for the free theory. Write your answer in terms of the two-point function Δ_0. (In class we showed that $\Delta_0(p) = -i/(p^2+m^2)$. But for this problem just treat it to be a function of spatial separation, do not try to evaluate it. In other words, your final answer should be something like $\Delta_0(x-y)\Delta_0(z-w)+\ldots$)

Make sure that the sum of the coefficients of the terms you find corresponds to the number of ways of pairing the $\delta/\delta J$'s.

Repeat this calculation by drawing the (three, identical looking except for permuting of labels) Feynman diagrams. (They will be disconnected.)

2.2 First order

Next we will find an expression for $G(x,y,z,w)$ to first order in λ. This requires two steps: finding $\delta^4 Z/\ldots$ to first order and $1/Z(J)$ to zero order, and finding $\delta^4 Z/\ldots$ to zero-order (last subsection) and $1/Z(J)$ to first order.

First, find an expression for $[\delta^4 Z(J)/\delta J(x)\delta J(y)\delta J(z)\delta J(w)]|_{J=0}$ to first order in λ. Do this by expanding the exponent containing λ, in Eq. (6), to first order in λ. You will now have to do a variation with respect to *eight* J's. I recommend that you *not* do this by taking eight derivatives of $\exp(-\int d^4u d^4v J(u)\Delta_0(u-v)J(v)/2)$, but instead that you use the results about pairing. You can either perform the pairing by hand, or by drawing the diagrams, whichever you find easier. But in either case, draw the diagrams *and* write the expressions; and group the expressions into three different "types" according to the topology of the graph. Make sure that the total coefficients of these terms correspond to the number of ways of pairing off the J derivatives.

Next, find the first-order-in-λ form for $1/Z(J)$. Since $Z(J)$ is in the denominator, you will have to write

$$\frac{1}{Z_0+\lambda Z_1} = \frac{1}{Z_0} - \frac{\lambda Z_1}{Z_0^2} \,. \tag{8}$$

The product of the order-λ term here, and the λ^0 term in $\partial^4 Z(J)/\partial J(x)\ldots$ which you found before, is also a term contributing at order λ to the 4-point correlation function.

Add these two types of order-λ contributions. You should find that the contributions from expanding the denominator *cancel* one of the three types of graphs you found by expanding the numerator, leaving the other two distinct types.

Argue that the terms which caneled off between numerator and denominator correspond precisely to the diagrams containing *vacuum bubbles*. However, we do *not* find only connected diagrams. Which diagrams are not connected?