Physics 610 Homework 5

Due Wednesday 24 October 2012

1 High-order Feynman diagrams

Consider the theory of one real scalar field ϕ with Lagrangian density

$$\mathcal{L}[\phi, \partial_\mu \phi] = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 \,. \tag{1}$$

1.1 6 external legs

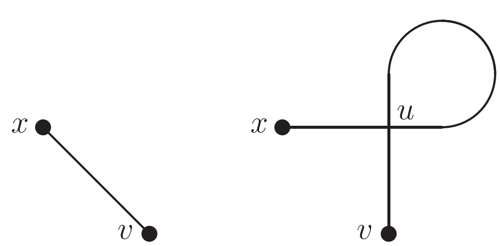
In class we considered the scattering of two scalars ϕ into two scalars ϕ, which involved investigating the four-point function $\langle 0|\, \mathbf{T}\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big)\, |0\rangle$. It is also possible for the final state to contain more than two particles. The simplest case is that it contains 4 final state particles; this requires evaluating the 6-point correlation function

$$\langle 0|\, \mathbf{T}\Big(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi(x_5)\phi(x_6)\Big)\, |0\rangle.$$

Draw the simplest connected Feynman diagram contributing to this correlation function which you can find. Now many occurrences of λ does it involve? Therefore, how many powers of λ will occur in the *rate* for this process (remembering that the rate involves the square of the matrix element $|\mathcal{M}|^2$)? If λ is small, do you expect that the rate for this process will be smaller or larger than the rate for the scattering process with two final state ϕ particles (assuming both are energetically allowed)?

1.2 Two-point function

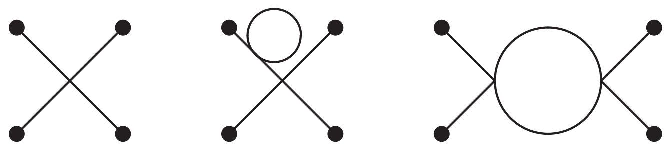
The simplest diagram contributing to the two-point function $\langle 0|\, \mathbf{T}\Big(\phi(x)\phi(v)\Big)\, |0\rangle$ and the first order *connected* contribution are given by the diagrams



Write the (position-space) expression associated with each graph. Name the sum of these expressions $-i\Delta(x-v)$, so $-i\Delta(x-v) = -i\Delta_0(x-v) + \ldots$.

1.3 Four-point function

The lowest-order connected diagram and two of the next-order connected diagrams for the four-point function (relevant in scattering) are



Label the external points x, y, z, w. Label the "main" cross point in the first two diagrams v, the extra cross-point in the middle diagram u, and the two cross-points (vertices) in the final diagram u, v.

First, find the symmetry factor for each diagram.

Second, figure out how many "very similar" diagrams (the same except for the assignments of external legs) there are for the middle and final diagrams.

Third, write an expression for each diagram (in position space), in terms of $\int d^4v d^4u$, $(-i\lambda)$ and the propagator $-i\Delta_0(x-v)$ etc.

Fourth: consider the first (simplest) diagram. Suppose you replace each appearance of the propagator $\Delta_0(x-v)$ with $\Delta(x-v)$ found in the last subsection. Since each $\Delta(x-v)$ is the sum of an order-λ^0 and an order-λ^1 piece, the resulting expression contains pieces of order λ^1, λ^2, and so forth. Find the λ^2 term which arises from the λ term in $\Delta(x-v)$, and show that it reproduces precisely the middle diagram. Argue that the role of the middle diagram and its "very similar" partners is to replace the leading-order propagators Δ_0 with the λ-improved propagators Δ.

Fifth: transform your expression for the final (rightmost) diagram into momentum space; take the external momenta to be p_1, p_2, p_3, p_4. Show that the energy-momentum conserving δ-functions associated with the two vertices (u, v integrations) can be rewritten as one δ-function forcing all the external momenta to sum to zero, plus another δ-function which performs one of the two remaining momentum integrals. Reduce your momentum-space expression for the diagram to something with only a single unperformed momentum $\int d^4q$ integral. *DO NOT* try to perform that integral. (We will return to that problem late in the term.)

2 Spinor identities

Here we clear up a bunch of properties of spinors and their transformation matrices which we claimed were true in class.

2.1 Pauli matrices

First (really easy), show that the Pauli matrices σ_i,

$$\sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

are Hermitian and traceless and satisfy the commutation relations

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}\,. \tag{3}$$

Next, show that the epsilon-matrix

$$\epsilon \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4}$$

obeys $\epsilon^2 = -1$ and that

$$-\sigma_i^*\epsilon = \epsilon\sigma_i \qquad \text{and} \qquad -\epsilon\sigma_i^* = \sigma_i\epsilon\,. \tag{5}$$

2.2 Left-handed spinors

Write $r_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk}$ so $\vec{r}$ gives the axis $\hat{r}$ and angle $|\vec{r}|$ of a rotation; and write $b_i = \omega_{i0}$ such that $\vec{b}$ gives the direction $\hat{b}$ and arctanh of the velocity $|\vec{b}|$ for a boost. In this notation the unitary operator corresponding to a Lorentz transformation is

$$U(\omega) = U(\vec{r}, \vec{b}) = \exp\Big(i(r_i\hat{J}_i + b_i\hat{K}_i)\Big)\,. \tag{6}$$

Take the expressions from lecture or from the book and show that

$$L_a{}^b(\omega) = \exp\left(\frac{i\sigma_i}{2}(r_i + ib_i)\right)\,. \tag{7}$$

If ψ_a transforms under Lorentz transform as

$$\psi_a \to L_a{}^b\psi_b \tag{8}$$

with $L_a{}^b$ given above, show that $\psi^\dagger_{\dot{a}} \equiv (\psi_a)^\dagger$ transforms as

$$\psi^\dagger_{\dot{a}} \to R_{\dot{a}}{}^{\dot{b}}\psi^\dagger_{\dot{b}} \qquad \text{with} \qquad R_{\dot{a}}{}^{\dot{b}} \equiv \exp\left(\frac{-i\sigma_i^*}{2}(r_i - ib_i)\right)\,. \tag{9}$$

Then show that $\psi^{\dagger\dot{a}} \equiv \epsilon^{\dot{a}\dot{b}}\psi^{\dagger}_{\dot{b}}$ transforms according to

$$\psi^{\dagger\dot{a}} \to R^{\dot{a}}{}_{\dot{b}}\psi^{\dagger\dot{b}} \qquad \text{with} \qquad R^{\dot{a}}{}_{\dot{b}} \equiv \exp\left(\frac{i\sigma_i}{2}(r_i - ib_i)\right) . \tag{10}$$

Therefore the (upper, dotted) index transforms like the (lower, undotted) left-handed index except that the sign on the boost is reversed.

2.3 Antisymmetric tensor

Show that

$$\epsilon^{ab}\psi_a\psi_b \to \epsilon^{ab}L_a{}^cL_b{}^d\psi_c\psi_d = \epsilon^{ab}\psi_a\psi_b\,, \qquad \text{that is,} \qquad \epsilon^{ab}L_a{}^cL_b{}^d = \epsilon^{ab} \tag{11}$$

for any $L_a{}^c$ of the form shown in Eq. (7). Therefore the combination $\epsilon^{ab}\psi_a\psi_b$ is a Lorentz scalar. Show similarly that

$$\epsilon_{\dot{a}\dot{b}}\psi^{\dagger\dot{a}}\psi^{\dagger\dot{b}} \tag{12}$$

is also a Lorentz scalar.

2.4 Mix of dotted and undotted

Define

$$\sigma^{\mu}_{a\dot{a}} \equiv \left(\mathbf{1}\,,\,\vec{\sigma}\right) \tag{13}$$

with $\mathbf{1}$ the 2×2 identity matrix and $\vec{\sigma}$ the Pauli matrices. Show to linear order in $\omega_{\mu\nu}$ that, under Lorentz transform, the combination

$$\sigma^{\mu}_{a\dot{a}}\psi^a\psi^{\dagger\dot{a}} \tag{14}$$

(where $\psi^a \equiv \epsilon^{ab}\psi_b$ as usual) transforms to

$$\begin{aligned} \sigma^{\mu}_{a\dot{a}}\psi^a\psi^{\dagger\dot{a}} &\to \sigma^{\mu}_{a\dot{a}}L^a{}_bR_{\dot{a}}{}^{\dot{b}}\psi^b\psi^{\dagger\dot{b}} \\ &= \Lambda^{\mu}{}_{\nu}\sigma^{\nu}_{a\dot{a}}\psi^a\psi^{\dagger\dot{a}}\,. \end{aligned} \tag{15}$$

That is, show that, at linear order in ω,

$$\sigma^{\mu}_{a\dot{a}}L^a{}_bR_{\dot{a}}{}^{\dot{b}} = \Lambda^{\mu}{}_{\nu}\sigma^{\nu}_{b\dot{b}}\,. \tag{16}$$

Therefore one lefthanded and one righthanded spinor combine into a 4-vector.