Physics 610 Homework 6

Due Wednesday 31 October 2012

1 Fermionic harmonic oscillator

Free scalar field theories reduce to a product of harmonic oscillators. Fermionic free theories reduce to a product of fermionic harmonic oscillators. These are actually simpler than the SHO, but you may not be familiar with them, so this problem forces you to mess with their properties.

For the fermionic oscillator, there is an operator b together with its Hermitian conjugate $b^\dagger$, which satisfy *anti*commutation relations,

$$\Big\{b\,,\,b\Big\} = 0 = \Big\{b^\dagger\,,\,b^\dagger\Big\}\,, \qquad \Big\{b\,,\,b^\dagger\Big\} = 1 \tag{1}$$

where as usual $\Big\{c\,,\,d\Big\} = cd + dc$. The Hamiltonian is $H = \frac{\omega}{2}(b^\dagger b - bb^\dagger)$.

Show that there must be a state annihilated by b. Hint: what is b^2? Name this (properly normalized) state $|0\rangle$.

Argue that $b^\dagger|0\rangle$ cannot vanish. Call this state $|1\rangle$. Show that it is properly normalized.

Now the punchline: show that, provided that all possible operators must be composed of products of b and $b^\dagger$, that $|0\rangle$ and $|1\rangle$ are the *only* two linearly independent states in the Hilbert space. That is, the operators $b, b^\dagger$ describe a 2-state system. (Hint: build all independent operators; there should only be 4. Show that the two states you found are closed under these operators.)

What is the energy of each state? Does the ground state have positive or negative energy?

2 Dirac equation and electron g factor

Verify that the Dirac Lagrangian density

$$\mathcal{L}[\Psi] = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \tag{2}$$

is invariant under the symmetry $\Psi \to e^{-i\theta}\Psi$, and $\bar{\Psi} \to e^{i\theta}\bar{\Psi}$ (which follows from complex conjugation). Find the Nöther current associated with this symmetry. (Careful: you have to sum over fields treating $\bar{\Psi}$ and Ψ as independent!)

Add to the Dirac Lagrangian a term $eA_\mu J^\mu$, with J^μ the Nöther current for the symmetry you just explored, and A^μ some spacetime-dependent 4-vector (which may be external or may represent another field). Show that this addition does not change the Nöther current you found. Then show that the Lagrangian can be written as

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu(\partial_\mu - ieA_\mu) - m)\Psi\,. \tag{3}$$

Derive the associated Euler-Lagrange equation, which should be

$$(i\not{\partial} + e\not{A} - m)\Psi = 0\,. \tag{4}$$

Multiply by $(-i\not{\partial} - e\not{A} - m)$. REMEMBER that the γ matrices do not all commute; show that the equation obeyed by ψ is

$$\left(p^2 + m^2 - eF_{\mu\nu}S^{\mu\nu}\right)\Psi = 0 \tag{5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $p_\mu = i\partial_\mu + eA_\mu$ is the kinetic momentum. Here $S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu\,,\,\gamma^\nu\right]$ as usual.

Now recall that Lorentz transformations are carried out by $S_{\mu\nu} = \frac{i}{4}\left[\gamma_\mu\,,\,\gamma_\nu\right]$, with

$$J_i = \frac{\epsilon_{ijk}}{2}S_{jk} \tag{6}$$

the angular momentum generator, and that the magnetic field is

$$B_i = \frac{\epsilon_{ijk}}{2}F_{jk}\,. \tag{7}$$

Rewrite the extra interaction term for the case of magnetic fields as an angular momentum (spin) dot magnetic interaction.

In the nonrelativistic limit, $m \gg \vec{p}, B$, what is the energy shift to an electron in the presence of a magnetic field? This shift is usually parametrized as

$$-\frac{ge}{2m}\vec{s}\cdot\vec{B} \tag{8}$$

with g a constant to be determined, and $\vec{s}$ the spin operator, $\vec{\sigma}/2$ for a spin-$\frac{1}{2}$ particle. What value do you find for g?

3 Gamma matrix identities

Using only that the γ matrices are 4×4 matrices satisfying the Clifford algebra, and using the definition

$$\not{a} \equiv \gamma^\mu a_\mu\,,$$

verify the following:

$$\not{k}\not{k} = -k^2 \tag{9}$$

$$\not{k}\not{p}\not{k} = -2p \cdot k\not{k} + k^2\not{p} \tag{10}$$

$$\gamma^\mu\gamma_\mu = -4 \tag{11}$$

$$\gamma^\mu\not{k}\gamma_\mu = 2\not{k} \tag{12}$$

$$\gamma^\mu\not{p}\not{k}\gamma_\mu = 4p \cdot k \tag{13}$$

$$\gamma^\mu\not{p}\not{k}\not{q}\gamma_\mu = 2\not{q}\not{k}\not{p}\,. \tag{14}$$

Hint: DO NOT multiply any 4×4 matrices to do this problem! Just use repeatedly that $AB = \left\{A\,,\,B\right\} - BA$ and recycle each identity as you prove successive ones.